

On the nonstationary Stokes system in a cone

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Abstract

The authors consider the Dirichlet problem for the nonstationary Stokes system in a three-dimensional cone. They obtain existence and uniqueness results for solutions in weighted Sobolev spaces and prove a regularity assertion for the solutions.

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Introduction

The present paper deals with the Dirichlet problem for the nonstationary Stokes system in a three-dimensional cone K . This means, we consider the problem

$$\frac{\partial u}{\partial t} - \Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } K \times (0, \infty), \quad (1)$$

$$u(x, t) = 0 \quad \text{for } x \in \partial K, \quad t > 0, \quad u(x, 0) = 0 \quad \text{for } x \in K. \quad (2)$$

The goal of the paper is to prove the existence and uniqueness of solutions in weighted Sobolev spaces and a regularity assertion for the solutions. A theory for the heat equation in domains with conical points and edges was established in a number of papers in the last 30 years, see [2], [7], [10], [11], [16], [18], [19]. This theory involves in particular existence and uniqueness results for solutions in weighted Sobolev and Hölder spaces, regularity assertions and the asymptotics of the solutions near vertices and edges. A class of general parabolic problems in a cone was studied in [4], [5], [6]. However, this class of problems does not include the Stokes system. Although the stationary Stokes system in domains with conical points and edges is well-studied (see, e. g., [3], [14], [15]), there is still no theory for the nonstationary Stokes system in domains with singular boundary points. The present paper is a first step in developing such a theory.

An essential part of the paper (Sections 1 and 2) consists of the investigation of the parameter-depending problem

$$s \tilde{u} - \Delta \tilde{u} + \nabla \tilde{p} = \tilde{f}, \quad -\nabla \cdot \tilde{u} = \tilde{g} \quad \text{in } K, \quad \tilde{u} = 0 \quad \text{on } \partial K, \quad (3)$$

which arises after the Laplace transformation with respect to the time t . In Section 1, we prove that this problem has a uniquely determined variational solution $(\tilde{u}, \tilde{p}) \in E_\beta^1(K) \times (V_\beta^0(K) + V_\beta^1(K))$ if $\operatorname{Re} s \geq 0$, $s \neq 0$ and $|\beta|$ is sufficiently small. Here $V_\beta^l(K)$ denotes the weighted Sobolev space of all functions (vector-functions) with finite norm

$$\|u\|_{V_\beta^l(K)} = \left(\int_K \sum_{|\alpha| \leq l} r^{2(\beta-l+|\alpha|)} |\partial_x^\alpha u(x)|^2 dx \right)^{1/2}, \quad (4)$$

while $E_\beta^l(K)$ is the weighted Sobolev space with the norm

$$\|u\|_{E_\beta^l(K)} = \left(\int_K \sum_{|\alpha| \leq l} (r^{2\beta} + r^{2(\beta-l+|\alpha|)}) |\partial_x^\alpha u(x)|^2 dx \right)^{1/2}, \quad (5)$$

$r = r(x)$ denotes the distance of the point x from the vertex of the cone.

The goal of Section 2 is to prove the existence and uniqueness of solutions $(\tilde{u}, \tilde{p}) \in E_\beta^2(K) \times V_\beta^1(K)$ of the parameter-depending problem in the case $\operatorname{Re} s \geq 0$, $s \neq 0$. Note that the problem (3) is not

elliptic with parameter in the sense of [1]. Therefore, the results concerning general parabolic problems in a cone which were obtained in [4, 5, 6] are not applicable to our problem.

We get two β -intervals for which we have an existence and uniqueness result in the space $E_\beta^2(K) \times V_\beta^1(K)$, namely the intervals

$$\frac{1}{2} - \lambda_1^+ < \beta < \frac{1}{2} \quad \text{and} \quad \frac{1}{2} < \beta < \min\left(\mu_2^+ + \frac{1}{2}, \lambda_1^+ + \frac{3}{2}\right) \quad (6)$$

(see Theorems 2.2 and 2.3). Here, λ_1^+ and μ_2^+ are positive numbers depending on the cone. More precisely, λ_1^+ is the smallest positive eigenvalues of the operator pencil $\mathcal{L}(\lambda)$ generated by the Dirichlet problem for the stationary Stokes system, while μ_2^+ is the smallest positive eigenvalue of the operator pencil $\mathcal{N}(\lambda)$ generated by the Neumann problem for the Laplacian, respectively. We show that the above inequalities for β are sharp.

We also prove a regularity assertion for the solutions of the problem (3), i. e., we answer the question under which conditions a solution $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ belongs to the space $E_\gamma^2(K) \times V_\gamma^1(K)$. In the case $\beta > \gamma$, we get a condition concerning the eigenvalues of the pencil $\mathcal{L}(\lambda)$, while the eigenvalues of the pencil $\mathcal{N}(\lambda)$ must be considered if $\beta < \gamma$ (see Lemma 2.9). The appearance of two different operator pencil is one important feature of the solvability and regularity theory for the problem (3). In analogous results for parameter-elliptic problems (see [4, 5]), only the eigenvalues of one operator pencil play a role. A further important difference with the parameter-elliptic case is the use of different function spaces. In the parameter-elliptic case, all functions (solutions and right-hand sides) belong to the spaces E_β^l . When considering the problem (3), we seek solutions $(\tilde{u}, \tilde{p}) \in E_\beta^2(K) \times V_\beta^1(K)$, while the right-hand sides \tilde{f}, \tilde{g} belong to the spaces $E_\beta^0(K)$ and $X_\beta^1(K) = V_\beta^1(K) \cap (V_{-\beta}^1(K))^*$, respectively.

In Section 3, we apply the results concerning the parameter-depending problem and obtain the following existence and uniqueness result for solutions of the problem (1), (2): If the data f and g belong to corresponding weighted Sobolev spaces, β lies in one of the intervals (6) and $\int_K g(x, t) dx = 0$ for almost all t if $\beta > 1/2$, then there exists a unique solution (u, p) of the problem (1), (2) such that

$$u \in L_2(\mathbb{R}_+, V_\beta^2(K)), \quad \partial_t u \in L_2(\mathbb{R}_+, V_\beta^0(K)), \quad p \in L_2(\mathbb{R}_+, V_\beta^1(K)).$$

Moreover, a regularity assertion for this solution is proved.

1 Weak solutions of the parameter-dependent problem

Let s be an arbitrary complex number, $\operatorname{Re} s \geq 0$. In this section, we prove the existence and uniqueness of weak solutions of the boundary value problem

$$s u - \Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } K, \quad u = 0 \quad \text{on } \partial K \setminus \{0\} \quad (7)$$

in weighted Sobolev spaces.

1.1 Weighted Sobolev spaces on the cone

Let $K = \{x \in \mathbb{R}^3 : \omega = x/|x| \in \Omega\}$ be a cone with vertex at the origin. Here, Ω is a subdomain of the unit sphere S^2 with smooth (of class C^2) boundary $\partial\Omega$. For nonnegative integer l and real β , we define the weighted Sobolev spaces $V_\beta^l(K)$ and $E_\beta^l(K)$ as the sets of all functions (or vector functions) with finite norms (4) and (5), respectively. Note that the spaces $V_\beta^l(K)$ and $E_\beta^l(K)$ can be also defined as the closures of $C_0^\infty(\overline{K} \setminus \{0\})$ with respect to the above norms. The space $E_\beta^l(K)$ can be also equipped with the equivalent norm

$$\|u\| = \left(\int_K r^{2\beta} \sum_{|\alpha|=l} |\partial_x^\alpha u(x)|^p dx + \int_K (r^{2\beta} + r^{2(\beta-l)}) |u(x)|^p dx \right)^{1/2}.$$

In the case $l \geq 1$, the trace spaces for $V_\beta^l(K)$ and $E_\beta^l(K)$ on $\partial K \setminus \{0\}$ are denoted by $V_\beta^{l-1/2}(\partial K)$ and $E_\beta^{l-1/2}(\partial K)$, respectively. If $l \leq 0$, then $V_\beta^{l-1/2}(\partial K)$ denotes the dual space of $V_{-\beta}^{-l+1/2}(\partial K)$. Furthermore, we define $\mathring{V}_\beta^1(K)$ and $\mathring{E}_\beta^1(K)$ as the spaces of all functions $u \in V_\beta^1(K)$ and $u \in E_\beta^1(K)$,

respectively, which are zero on $\partial K \setminus \{0\}$. The dual spaces of $\mathring{V}_\beta^1(K)$ and $\mathring{E}_\beta^1(K)$ are denoted by $V_{-\beta}^{-1}(K)$ and $E_{-\beta}^{-1}(K)$, respectively. Since

$$\int_K r^{2\beta-2} |u(x)|^2 dx \leq c \int_K r^{2\beta} |\nabla u(x)|^2 dx$$

for $u \in C_0^\infty(K)$, the norm

$$\|u\| = \left(\int_K r^{2\beta} (|u|^2 + |\nabla u|^2) dx \right)^{1/2}$$

is equivalent to the $E_\beta^1(K)$ -norm in $\mathring{E}_\beta^1(K)$. Finally, we mention that there exists a constant c such that

$$\int_{\partial K} r^{2(\beta-l)+1} |u(x)|^2 dx \leq c \|u\|_{V_\beta^{l-1/2}(\partial K)}^2 \quad (8)$$

for all $u \in V_\beta^{l-1/2}(\partial K)$, $l \geq 1$, and

$$\int_{\partial K} (r^{2\beta} + r^{2(\beta-l)+1}) |u(x)|^2 dx \leq c \|u\|_{E_\beta^{l-1/2}(\partial K)}^2 \quad (9)$$

for all $u \in E_\beta^{l-1/2}(\partial K)$, $l \geq 1$ (cf. [13, Lemmas 1.4 and 1.5]).

1.2 A property of the operator div

It is obvious that the operator div realizes a linear and continuous mapping

$$\mathring{E}_0^1(K) \rightarrow L_2(K) \cap (V_0^1(K))^*$$

Our goal is to show that this operator is surjective. To this end, we introduce the following operator pencil \mathcal{L} generated by the stationary Stokes system in the cone K . For every complex λ , we define the operator $\mathcal{L}(\lambda)$ as the mapping

$$\begin{aligned} \mathring{W}^1(\Omega) \times L_2(\Omega) &\ni \begin{pmatrix} u \\ p \end{pmatrix} \\ &\rightarrow \begin{pmatrix} r^{2-\lambda} (-\Delta r^\lambda u(\omega) + \nabla r^{\lambda-1} p(\omega)) \\ -r^{1-\lambda} \nabla \cdot (r^\lambda u(\omega)) \end{pmatrix} \in W^{-1}(\Omega) \times L_2(\Omega), \end{aligned}$$

where $r = |x|$ and $\omega = x/|x|$. The properties of the pencil \mathcal{L} are studied, e.g., in [9]. In particular, it is known that the eigenvalues of this pencil in the strip $-2 \leq \operatorname{Re} \lambda \leq 1$ are real and that the numbers 1 and -2 are always eigenvalues. If Ω is contained in a half-sphere, then these numbers are the only eigenvalues in the interval $[-2, 1]$ (cf. [9, Theorem 5.5.5]). We use this fact for the proof of the following lemma.

Lemma 1.1 *For arbitrary $g \in L_2(K) \cap (V_0^1(K))^*$, there exists a vector function $v \in \mathring{E}_0^1(K)$ such that $-\nabla \cdot v = g$ in K and*

$$\|v\|_{E_0^1(K)} \leq c \left(\|g\|_{L_2(K)} + \|g\|_{(V_0^1(K))^*} \right). \quad (10)$$

Here, c is a constant independent of g .

P r o o f. Assume first that Ω is contained in a half-sphere. We denote the operator $(u, p) \rightarrow (f, g)$ of the boundary value problem

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } K, \quad u = 0 \quad \text{on } \partial K \setminus \{0\} \quad (11)$$

by A_0 . Furthermore, we introduce the spaces

$$\begin{aligned} X_1 &= \mathring{V}_0^1(K) \times L_2(K), \quad Y_1 = V_0^{-1}(K) \times L_2(K), \\ X_2 &= (V_0^2(K) \cap \mathring{V}_{-1}^1(K)) \times V_0^1(K), \quad Y_2 = L_2(K) \times V_0^1(K). \end{aligned}$$

By [15, Theorem 10.2.11], the operator A_0 realizes an isomorphism $X_1 \rightarrow Y_1$, i.e., for every pair $(f, g) \in Y_1$, there exists a uniquely determined vector function $(u, p) \in X_1$ such that

$$b_0(u, v) - \int_K p \nabla \cdot v \, dx = \langle f, v \rangle \quad \text{for all } v \in \mathring{V}_0^1(K), \quad -\nabla \cdot u = g \quad \text{in } K,$$

where $b_0(u, v)$ is the bilinear form

$$b_0(u, v) = \sum_{j=1}^3 \int_K \nabla u_j \cdot \nabla v_j \, dx$$

(here u_j, v_j are the components of the vector functions u and v , respectively). Furthermore by [15, Theorem 10.3.2], the operator A_0 realizes an isomorphism $X_2 \rightarrow Y_2$. From [15, Theorem 10.6.9] it follows that every solution $(u, p) \in X_1$ of the equation

$$A_0(u, p) = (f, g)$$

belongs to the space X_2 if $(f, g) \in Y_1 \cap Y_2$. Consequently, the operator A_0 is also an isomorphism from $X_1 \cap X_2$ onto $Y_1 \cap Y_2$ and from $X_1 + X_2$ onto $Y_1 + Y_2$, and the adjoint operator A_0^* realizes an isomorphism $Y_1^* \cap Y_2^* \rightarrow X_1^* \cap X_2^*$. This means that for every $(f, g) \in X_1^* \cap X_2^*$, there exists a uniquely determined element $(v, q) \in Y_1^* \cap Y_2^*$ such that

$$\begin{aligned} (u, f)_K + (p, g)_K &= (-\Delta u + \nabla p, v)_K - (\nabla \cdot u, q)_K \quad \text{for all } (u, p) \in X_2, \\ (u, f)_K + (p, g)_K &= b_0(u, v) - (p, \nabla \cdot v)_K - (\nabla \cdot u, q)_K \quad \text{for all } (u, p) \in X_1. \end{aligned}$$

Here $(\cdot, \cdot)_K$ denotes the extension of the $L_2(K)$ scalar product to pairs from $X \times X^*$. Moreover, there exists a constant c independent of f, g such that

$$\|(v, q)\|_{Y_1^* \cap Y_2^*} \leq c \|(f, g)\|_{X_1^* \cap X_2^*}.$$

In particular (for $f = 0, u = 0$), we conclude that for arbitrary $g \in L_2(K) \cap V_0^1(K)^*$, there exists a vector function $v \in \mathring{V}_0^1(K) \cap L_2(K) = \mathring{E}_0^1(K)$ such that

$$(p, g)_K = (\nabla p, v)_K \quad \text{for all } p \in V_0^1(K) \quad \text{and} \quad (p, g)_K = -(p, \nabla \cdot v)_K \quad \text{for all } p \in L_2(K).$$

Furthermore, v satisfies (10). This proves the assertion for the case when Ω is contained in a half-sphere.

In the contrary case, one can find subdomains $\Omega_j \subset \Omega$, $j = 1, \dots, n$, with smooth boundaries and smooth function χ_j on Ω such that $\Omega = \Omega_1 \cup \dots \cup \Omega_n$, Ω_j is contained in a half-sphere for every j ,

$$\text{supp } \chi_j \subset \overline{\Omega_j} \quad \text{and} \quad \sum_{j=1}^n \zeta_j = 1.$$

Let K_j be the cone consisting of all $x \in K$ such that $x/|x| \in \Omega_j$. By the first part of the proof, there exist vector functions $w_j \in \mathring{E}_0^1(K_j)$ such that $\nabla \cdot w_j(x) = -\chi_j(\omega) g(x)$ for $x \in K_j$ and

$$\|w_j\|_{\mathring{E}_0^1(K_j)} \leq c \left(\|\chi_j g\|_{L_2(K_j)} + \|\chi_j g\|_{(V_0^1(K_j))^*} \right)$$

Let v_j be the extensions of w_j by zero to the cone K . It is obvious that $v_j \in \mathring{E}_0^1(K)$ and that the sum $v = v_1 + \dots + v_n$ satisfies the equation $\nabla \cdot v = -g$ in K and the estimate (10). \square

1.3 Existence of variational solutions

In order to define variational solutions of the problem (7), we introduce the bilinear form

$$b_s(u, v) = \int_K \left(su \cdot v + \sum_{j=1}^3 \nabla u_j \cdot \nabla v_j \right) dx.$$

It is clear that the mappings

$$v \rightarrow b_s(u, v) \quad \text{and} \quad v \rightarrow \int_K p \nabla \cdot v \, dx$$

define linear and continuous functionals on $\mathring{E}_0^1(K)$ if $u \in \mathring{E}_0^1(K)$ and $p \in L_2(K) + V_0^1(K)$. By a *variational solution* of the problem (7), we mean a pair $(u, p) \in \mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K))$ satisfying the equations

$$b_s(u, v) - \int_K p \nabla \cdot v \, dx = \langle f, v \rangle \quad \text{for all } v \in \mathring{E}_0^1(K), \quad -\nabla \cdot u = g \quad \text{in } K, \quad (12)$$

where $f \in E_0^{-1}(K)$ and $g \in L_2(K) \cap (V_0^1(K))^*$ are given.

Theorem 1.1 *Suppose that $|s| = 1$, $\operatorname{Re} s \geq 0$, $f \in E_0^{-1}(K)$ and $g \in L_2(K) \cap (V_0^1(K))^*$. Then there exists a uniquely determined solution $(u, p) \in \mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K))$ of the problem (12). Furthermore,*

$$\|u\|_{E_0^1(K)} + \|p\|_{L_2(K) + V_0^1(K)} \leq c \left(\|f\|_{E_0^{-1}(K)} + \|g\|_{L_2(K)} + \|g\|_{(V_0^1(K))^*} \right)$$

with a constant c independent of f, g and s .

P r o o f. First, we prove the existence of solutions in the case $g = 0$. We introduce the subspace

$$H = \{u \in \mathring{E}_0^1(K) : \nabla \cdot u = 0 \text{ in } K\}$$

of $\mathring{E}_0^1(K)$ and denote by H^\perp its orthogonal complement in $\mathring{E}_0^1(K)$. Since

$$|b_s(u, \bar{u})| \geq \frac{1}{2} \|u\|_{E_0^1(K)}^2 \quad \text{for all } u \in E_0^1(K)$$

if $|s| = 1$, $\operatorname{Re} s \geq 0$, there exists a uniquely determined vector function $u \in H$ such that

$$b_s(u, v) = \langle f, v \rangle \quad \text{for all } v \in H, \quad \|u\|_{E_0^1(K)} \leq c \|f\|_{E_0^{-1}(K)}. \quad (13)$$

By Lemma 1.1, the operator div is an isomorphism from H^\perp onto $L_2(K) \cap (V_0^1(K))^*$, i.e., for every $q \in L_2(K) \cap (V_0^1(K))^*$, there exists a uniquely determined vector function $v = (-\operatorname{div})^{-1}q \in H^\perp$ such that $\nabla \cdot v = -q$ and

$$\|v\|_{E_0^1(K)} \leq c \left(\|q\|_{L_2(K)} + \|q\|_{(V_0^1(K))^*} \right).$$

We consider the functional

$$\ell(q) = \langle f, v \rangle - b_s(u, v) = \langle f, (-\operatorname{div})^{-1}q \rangle - b_s(u, (-\operatorname{div})^{-1}q) \quad (14)$$

on $L_2(K) \cap (V_0^1(K))^*$. Obviously,

$$\begin{aligned} |\ell(q)| &\leq c \left(\|f\|_{E_0^{-1}(K)} + \|u\|_{E_0^1(K)} \right) \|v\|_{E_0^1(K)} \\ &\leq c \|f\|_{E_0^{-1}(K)} \left(\|q\|_{L_2(K)} + \|q\|_{(V_0^1(K))^*} \right). \end{aligned}$$

This means that ℓ is continuous on $L_2(K) \cap (V_0^1(K))^*$ and there exists an element p of the dual space $L_2(K) + V_0^1(K)$ such that $\ell(q) = \int_K p q \, dx$ for all $q \in L_2(K) \cap (V_0^1(K))^*$. Then by (14), we have

$$b_s(u, v) + \int_K p q \, dx = \langle f, v \rangle$$

for $q \in L_2(K) \cap (V_0^1(K))^*$, $v = (-\operatorname{div})^{-1}q \in H^\perp$. This means that

$$b_s(u, v) - \int_K p \nabla \cdot v \, dx = \langle f, v \rangle \quad \text{for all } v \in H^\perp.$$

Using (13), we conclude that (u, p) is a solution of the problem (12) in the case $g = 0$.

Now let $g \in L_2(K) \cap (V_0^1(K))^*$ be an arbitrary function. By Lemma 1.1, there exists a function $w \in \mathring{E}_0^1(K)$ such that

$$-\nabla \cdot w = g \quad \text{in } K, \quad \|w\|_{E_0^1(K)} \leq c \left(\|g\|_{L_2(K)} + \|g\|_{(V_0^1(K))^*} \right).$$

Obviously the mapping

$$\mathring{E}_0^1(K) \ni v \rightarrow \langle f, v \rangle - b_s(w, v)$$

defines a continuous functional on $\mathring{E}_0^1(K)$. Therefore, by the first part of the proof, there exist functions $W \in \mathring{E}_0^1(K)$ and $p \in L_2(K) + V_0^1(K)$ such that

$$b_s(W, v) - \int_K p \nabla \cdot v \, dx = \langle f, v \rangle - b_s(w, v) \quad \text{for all } v \in \mathring{E}_0^1(K).$$

Then $(u, p) = (W + w, p)$ is a solution of the problem (12).

We prove the uniqueness of the solution. Let $(u, p) \in \mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K))$ be a solution of the problem (12) with $f = 0$ and $g = 0$. Then in particular, $u \in H$ and $b_s(u, \bar{u}) = 0$ what implies $u = 0$ and $\int_K p \nabla \cdot v \, dx = 0$ for all $v \in \mathring{E}_0^1(K)$. Using Lemma 1.1, we conclude that $(p, q)_K = 0$ for all $q \in L_2(K) \cap (V_0^1(K))^*$ and, consequently, $p = 0$. The proof of the theorem is complete. \square

As a consequence of the last theorem, we obtain the existence and uniqueness of variational solutions in the space $\mathring{E}_\beta^1(K) \times (V_\beta^0(K) + V_\beta^1(K))$ if $|\beta|$ is sufficiently small. Suppose that $f \in E_\beta^{-1}(K)$ and $g \in V_\beta^0(K) \cap (V_{-\beta}^1(K))^*$. Then $(u, p) \in \mathring{E}_\beta^1(K) \times (V_\beta^0(K) + V_\beta^1(K))$ is called a variational solution of the problem (7) if

$$b_s(u, v) - \int_K p \nabla \cdot v \, dx = \langle f, v \rangle \quad \text{for all } v \in \mathring{E}_{-\beta}^1(K), \quad -\nabla \cdot u = g \quad \text{in } K. \quad (15)$$

Corollary 1.1 *Suppose that $|s| = 1$, $\operatorname{Re} s \geq 0$, $f \in E_\beta^{-1}(K)$ and $g \in V_\beta^0(K) \cap (V_{-\beta}^1(K))^*$, where $|\beta|$ is sufficiently small. Then there exists a uniquely determined solution $(u, p) \in \mathring{E}_\beta^1(K) \times (V_\beta^0(K) + V_\beta^1(K))$ of the problem (15). This solution satisfies the estimate*

$$\|u\|_{E_\beta^1(K)} + \|p\|_{V_\beta^0(K) + V_\beta^1(K)} \leq c \left(\|f\|_{E_\beta^{-1}(K)} + \|g\|_{V_\beta^0(K)} + \|g\|_{(V_{-\beta}^1(K))^*} \right)$$

with a constant c independent of f, g and s .

P r o o f. Let $(U, P) = r^\beta (u, p)$. Obviously,

$$(u, p) \in \mathring{E}_\beta^1(K) \times (V_\beta^0(K) + V_\beta^1(K)) \Leftrightarrow (U, P) \in \mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K)).$$

Furthermore, (u, p) is a solution of (15) if and only if (U, P) satisfies

$$b_s(r^{-\beta}U, r^\beta V) - \int_K r^{-\beta}P \nabla \cdot r^\beta V \, dx = \langle F, V \rangle \quad \text{for all } V = r^{-\beta}v \in \mathring{E}_0^1(K), \quad (16)$$

$$-r^\beta \nabla \cdot (r^{-\beta}U) = G \quad \text{in } K, \quad (17)$$

where $F = r^\beta f \in E_0^{-1}(K)$ and $G = r^\beta g \in L_2(K) \cap (V_0^1(K))^*$. We denote the operator

$$\mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K)) \ni (U, P) \rightarrow (F, G) \in E_0^{-1}(K) \times (L_2(K) \cap (V_0^1(K))^*)$$

by A_β . By Theorem 1.1, the operator A_0 is an isomorphism. Since the operator $A_0 - A_\beta$ has a small norm for small $|\beta|$, it follows that A_β is an isomorphism if $|\beta|$ is sufficiently small. This proves the corollary. \square

2 Strong solutions of the problem (7)

Now, we are interested in solutions $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ of the problem (7).

2.1 The operator of the problem (7)

Obviously, the mapping

$$E_\beta^2(K) \times V_\beta^1(K) \ni (u, p) \rightarrow f = su - \Delta u + \nabla p \in E_\beta^0(K)$$

and the mapping $E_\beta^2(K) \ni u \rightarrow g = -\nabla \cdot u \in E_\beta^1(K)$ are continuous for arbitrary real β . Furthermore, for arbitrary $u \in E_\beta^2(K) \cap \mathring{E}_\beta^1(K)$ and $q \in V_{-\beta}^1(K)$, we get

$$\left| \int_K q \nabla \cdot u \, dx \right| = \left| \int_K u \cdot \nabla q \, dx \right| \leq \|u\|_{V_\beta^0(K)} \|q\|_{V_{-\beta}^1(K)} \leq \|u\|_{E_\beta^2(K)} \|q\|_{V_{-\beta}^1(K)}.$$

Thus, the mapping $u \rightarrow g = -\nabla \cdot u$ is continuous from $E_\beta^2(K) \cap \mathring{E}_\beta^1(K)$ into the space

$$X_\beta^1(K) = E_\beta^1(K) \cap (V_{-\beta}^1(K))^*.$$

and the operator of the problem (7) realizes a linear and continuous mapping from $(E_\beta^2(K) \cap \mathring{E}_\beta^1(K)) \times V_\beta^1(K)$ into $E_\beta^0(K) \times X_\beta^1(K)$ for arbitrary real β . We denote the operator

$$(E_\beta^2(K) \cap \mathring{E}_\beta^1(K)) \times V_\beta^1(K) \ni (u, p) \rightarrow (su - \Delta u + \nabla p, -\nabla \cdot u) \in E_\beta^0(K) \times X_\beta^1(K) \quad (18)$$

of this problem by A_β . Note that

$$\int_K r^{2\beta} |g|^2 \, dx \leq \|g\|_{(V_{-\beta}^1(K))^*} \|r^{2\beta} g\|_{V_{-\beta}^1(K)} \leq c \|g\|_{(V_{-\beta}^1(K))^*} \|g\|_{V_\beta^1(K)}.$$

for $g \in V_\beta^1(K) \cap (V_{-\beta}^1(K))^*$. Consequently, $V_\beta^0(K) \subset V_\beta^1(K) \cap (V_{-\beta}^1(K))^*$. This implies that

$$X_\beta^1(K) = V_\beta^1(K) \cap (V_{-\beta}^1(K))^*.$$

Lemma 2.1 Suppose that $u \in E_\beta^2(K) \cap \mathring{E}_\beta^1(K)$, $p \in V_\beta^1(K)$ and $A_\beta(u, p) = (f, g)$. If $0 \leq \beta \leq 1$, then (u, p) coincides with the variational solution of the problem (7) in the space $\mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K))$.

P r o o f. Suppose that $0 \leq \beta \leq 1$. Then

$$E_\beta^2(K) \cap \mathring{E}_\beta^1(K) \subset \mathring{E}_0^1(K) \quad \text{and} \quad V_\beta^1(K) \subset L_2(K) + V_0^1(K).$$

Furthermore, $E_\beta^0(K) \subset E_0^{-1}(K)$ and $X_\beta^1(K) \subset L_2(K) \cap (V_0^1(K))^*$ for $0 \leq \beta \leq 1$. Obviously, every solution $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ of (7) satisfies (12). This proves the lemma. \square

In particular, it follows from Theorem 1.1 and Lemma 2.1 that the kernel of A_β is trivial for $0 \leq \beta \leq 1$. Later, we will improve this result.

2.2 An a priori estimate for the solutions of the problem (7)

In order to prove a local estimate for the solutions of the problem (7), we employ the following lemma.

Lemma 2.2 Let \mathcal{D} be a bounded domain with smooth (of class C^2) boundary $\partial\mathcal{D}$. Suppose that $u \in W_2^2(\mathcal{D})$, $\nabla p \in L_2(\mathcal{D})$,

$$s u - \Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } \mathcal{D}, \quad u = 0 \quad \text{on } \partial\mathcal{D}.$$

Then

$$\|D^2 u\|_{L_2(\mathcal{D})} + |s| \|u\|_{L_2(\mathcal{D})} + \|\nabla p\|_{L_2(\mathcal{D})} \leq c \left(\|f\|_{L_2(\mathcal{D})} + \|g\|_{W_2^1(\mathcal{D})} + |s| \|g\|_{W_2^{-1}(\mathcal{D})} + \|u\|_{L_2(\mathcal{D})} \right) \quad (19)$$

with a constant independent of s . Here, $D^2 u$ denotes the vector of all second order derivatives of u .

P r o o f. The functions $v(x, t) = e^{st}u(x)$, $q(x, t) = e^{st}p(x)$ satisfy the equations

$$\begin{aligned} \partial_t v(x, t) - \Delta v(x, t) + \nabla q(x, t) &= e^{st}f(x), \quad -\nabla \cdot v(x, t) = e^{st}g(x) \quad \text{for } x \in \mathcal{D}, \\ v(x, t) &= 0 \quad \text{for } x \in \partial\mathcal{D}, \quad v(x, 0) = u(x) \quad \text{for } x \in \mathcal{D}. \end{aligned}$$

Consequently by [20, Theorem 3.1],

$$\begin{aligned} &\|D^2 v\|_{L_2(\mathcal{D} \times (0,1))} + \|\partial_t v\|_{L_2(\mathcal{D} \times (0,1))} + \|\nabla q\|_{L_2(\mathcal{D} \times (0,1))} \\ &\leq c \left(\|e^{st}f\|_{L_2(\mathcal{D} \times (0,1))} + \|e^{st}g\|_{L_2(0,1;W_2^1(\mathcal{D}))} + \|\partial_t(e^{st}g)\|_{L_2(0,1;W_2^{-1}(\mathcal{D}))} + \|u\|_{L_2(\mathcal{D})} \right). \end{aligned}$$

The result follows. \square

In the following, let ζ_ν be infinitely differentiable functions on $\mathbb{R}_+ = (0, \infty)$ with support in the interval $(2^{\nu-1}, 2^{\nu+1})$ such that

$$|\partial_r^j \zeta_\nu(r)| \leq 2^{-j\nu} \quad \text{for } j = 0, 1, \dots \quad \text{and} \quad \sum_{\nu=-\infty}^{+\infty} \zeta_\nu = 1.$$

Using the notation $r = |x|$, the functions $x \rightarrow \zeta_\nu(r)$ can be considered as functions in K .

Lemma 2.3 *Suppose that $(u, p) \in (E_\beta^2(K) \cap \mathring{E}_\beta^1(K)) \times V_\beta^1(K)$ is a solution of (7). Then*

$$\begin{aligned} &\|D^2(\zeta_\nu u)\|_{L_2(K)}^2 + |s|^2 \|\zeta_\nu u\|_{L_2(K)}^2 + \|\nabla(\zeta_\nu p)\|_{L_2(K)}^2 \\ &\leq c \left(\|f\|_{L_2(K_\nu)}^2 + \|\nabla(\zeta_\nu g)\|_{L_2(K_\nu)}^2 + 2^{-2\nu} \|\zeta_\nu g\|_{L_2(K_\nu)}^2 + |s|^2 \|\zeta_\nu g\|_{W_2^{-1}(K_\nu)}^2 \right. \\ &\quad \left. + 2^{-2\nu} \|\nabla u\|_{L_2(K_\nu)}^2 + 2^{-4\nu} \|u\|_{L_2(K_\nu)}^2 + 2^{-2\nu} \|p\|_{L_2(K_\nu)}^2 \right), \end{aligned} \quad (20)$$

where $K_\nu = \{x \in K : 2^{\nu-1} < |x| < 2^{\nu+1}\}$. The constant c in (20) is independent of ν and s .

P r o o f. Obviously,

$$s \zeta_\nu u - \Delta(\zeta_\nu u) + \nabla(\zeta_\nu p) = F, \quad -\nabla \cdot (\zeta_\nu u) = G \quad \text{in } K, \quad (21)$$

where $F_i = \zeta_\nu f_i - 2\nabla \zeta_\nu \cdot \nabla u_i - u_i \Delta \zeta_\nu + p \partial_{x_i} \zeta_\nu$ for $i = 1, 2, 3$ and $G = \zeta_\nu g - u \cdot \nabla \zeta_\nu$. We define

$$v(x) = \zeta_\nu(2^\nu r) u(2^\nu x), \quad q(x) = 2^\nu \zeta_\nu(2^\nu r) p(2^\nu x).$$

By (21), we have

$$2^{2\nu} s v - \Delta v + \nabla q = \Phi, \quad -\nabla \cdot v = \Psi \quad \text{in } K,$$

where $\Phi(x) = 2^{2\nu} F(2^\nu x)$ and $\Psi(x) = 2^\nu G(2^\nu x)$. Since v and q vanish outside the set $K_0 = \{x \in K : 1/2 < |x| < 2\}$, it follows from Lemma 2.2 that

$$\begin{aligned} &\|D^2 v\|_{L_2(K)} + 2^{2\nu} |s| \|v\|_{L_2(K)} + \|\nabla q\|_{L_2(K)} \\ &\leq c \left(\|\Phi\|_{L_2(K_0)} + \|\Psi\|_{W_2^1(K_0)} + 2^{2\nu} |s| \|\Psi\|_{W_2^{-1}(K_0)} + \|v\|_{L_2(K_0)} \right), \end{aligned} \quad (22)$$

where the constant c is independent of ν and s . One can easily check that

$$\|v\|_{L_2(K)} = 2^{-\nu n/2} \|\zeta_\nu u\|_{L_2(K)}, \quad \|D^2 v\|_{L_2(K)} = 2^{\nu(2-n/2)} \|D^2(\zeta_\nu u)\|_{L_2(K)}$$

and

$$\|\nabla q\|_{L_2(K)} = 2^{\nu(2-n/2)} \|\nabla(\zeta_\nu p)\|_{L_2(K)}.$$

Furthermore, we obtain the estimates

$$\begin{aligned} &\|\Phi\|_{L_2(K)} + \|\Psi\|_{W_2^1(K)} = 2^{\nu(2-n/2)} \left(\|F\|_{L_2(K)} + \|\nabla G\|_{L_2(K)} + 2^{-\nu} \|G\|_{L_2(K)} \right) \\ &\leq c 2^{\nu(2-n/2)} \left(\|\zeta_\nu f\|_{L_2(K)} + \|\nabla(\zeta_\nu g)\|_{L_2(K)} + 2^{-\nu} \|\zeta_\nu g\|_{L_2(K)} \right. \\ &\quad \left. + 2^{-\nu} \|\nabla u\|_{L_2(K_\nu)} + 2^{-2\nu} \|u\|_{L_2(K_\nu)} + 2^{-\nu} \|p\|_{L_2(K_\nu)} \right). \end{aligned}$$

Moreover,

$$\|\Psi\|_{W_2^{-1}(K_0)} = 2^{-\nu n/2} \|G\|_{W_2^{-1}(K_\nu)} \leq c 2^{-\nu n/2} \left(\|\zeta_\nu g\|_{W_2^{-1}(K_\nu)} + 2^{-\nu} \|u\|_{W_2^{-1}(K_\nu)} \right)$$

Using the equality $su = f + \Delta u - \nabla p$ and the estimate $\|f\|_{W_2^{-1}(K_\nu)} \leq 2^\nu \|f\|_{L_2(K_\nu)}$, we get

$$\|\Psi\|_{W_2^{-1}(K_0)} \leq c 2^{-\nu n/2} \|\zeta_\nu g\|_{W_2^{-1}(K_\nu)} + c 2^{-\nu(1+n/2)} |s|^{-1} \left(2^\nu \|f\|_{L_2(K_\nu)} + \|\nabla u\|_{L_2(K_\nu)} + \|p\|_{L_2(K_\nu)} \right).$$

Thus, the estimate (22) implies (20). \square

Using the estimate (20), one can easily prove the following lemma.

Lemma 2.4 *Suppose that $u \in W_2^2(\mathcal{D})$, $p \in W_2^1(\mathcal{D})$ for every bounded subdomain $\mathcal{D} \subset K$, $\overline{\mathcal{D}} \subset \overline{K} \setminus \{0\}$, and that $u \in V_{\beta-1}^1(K)$, $p \in V_{\beta-1}^0(K)$. If (u, p) satisfies (7) with $|s| = 1$ and with data $f \in V_\beta^0(K)$, $g \in X_\beta^1(K)$, then $u \in E_\beta^2(K)$, $p \in V_\beta^1(K)$ and*

$$\|u\|_{E_\beta^2(K)} + \|p\|_{V_\beta^1(K)} \leq c \left(\|f\|_{V_\beta^0(K)} + \|g\|_{X_\beta^1(K)} + \|u\|_{V_{\beta-1}^1(K)} + \|p\|_{V_{\beta-1}^0(K)} \right).$$

P r o o f. Multiplying (20) by $2^{2\nu\beta}$, we get

$$\|\zeta_\nu u\|_{E_\beta^2(K)}^2 + \|\zeta_\nu p\|_{V_\beta^1(K)}^2 \leq c \left(\|r^\beta f\|_{L_2(K_\nu)}^2 + \|\zeta_\nu g\|_{X_\beta^1(K)}^2 + \|\eta_\nu u\|_{V_{\beta-1}^1(K)}^2 + \|\eta_\nu p\|_{V_{\beta-1}^0(K_\nu)}^2 \right) \quad (23)$$

for every ν , where $\eta_\nu = \zeta_{\nu-1} + \zeta_\nu + \zeta_{\nu+1} = 1$ on K_ν . The constant c in (23) is independent of ν . Using the inequalities

$$\begin{aligned} & \sum_{\nu=-\infty}^{+\infty} \left(\|r^\beta f\|_{L_2(K_\nu)}^2 + \|\zeta_\nu g\|_{X_\beta^1(K)}^2 + \|\eta_\nu u\|_{V_{\beta-1}^1(K)}^2 + \|\eta_\nu p\|_{V_{\beta-1}^0(K_\nu)}^2 \right) \\ & \leq c \left(\|f\|_{V_\beta^0(K)}^2 + \|g\|_{X_\beta^1(K)}^2 + \|u\|_{V_{\beta-1}^1(K)}^2 + \|p\|_{V_{\beta-1}^0(K)}^2 \right) \end{aligned}$$

and

$$\|u\|_{E_\beta^2(K)}^2 + \|p\|_{V_\beta^1(K)}^2 \leq c \sum_{\nu=-\infty}^{+\infty} \left(\|\zeta_\nu u\|_{E_\beta^2(K)}^2 + \|\zeta_\nu p\|_{V_\beta^1(K)}^2 \right),$$

we obtain the desired result. \square

2.3 An auxiliary problem for the function p

Suppose that $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ is a solution of the problem (7). Multiplying the equation $su - \Delta u + \nabla p = f$ by ∇q and integrating, we obtain

$$\int_K \nabla p \cdot \nabla q \, dx = \langle F, q \rangle \quad \text{for all } q \in V_{-\beta}^1(K), \quad (24)$$

where

$$\langle F, q \rangle = \int_K ((f + \Delta u) \cdot \nabla q - s g q) \, dx.$$

The following lemma allows us to use another representation of the functional F .

Lemma 2.5 *Suppose that $u \in E_\beta^2(K)$ and $q \in V_{-\beta}^1(K) \cap V_{1-\gamma}^2(K)$, where $\beta \leq \gamma \leq \beta + 1/2$. Then*

$$\int_K \Delta u \cdot \nabla q \, dx = - \int_K \nabla g \cdot \nabla q \, dx + \int_{\partial K} \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \left(n_j \frac{\partial q}{\partial x_i} - n_i \frac{\partial q}{\partial x_j} \right) d\sigma,$$

where $g = -\nabla \cdot u$.

P r o o f. Suppose that $q \in C_0^\infty(\overline{K} \setminus \{0\})$. Then

$$\begin{aligned} \int_K (\Delta u + \nabla g) \cdot \nabla q \, dx &= \int_K \left(\sum_{i=1}^3 \Delta u_i \frac{\partial q}{\partial x_i} - \sum_{j=1}^3 \frac{\partial(\nabla \cdot u)}{\partial x_j} \frac{\partial q}{\partial x_j} \right) dx \\ &= \int_K \sum_{i,j=1}^3 \left(\frac{\partial^2 u_i}{\partial x_j^2} \frac{\partial q}{\partial x_i} - \frac{\partial^2 u_i}{\partial x_i \partial x_j} \frac{\partial q}{\partial x_j} \right) dx = \int_K \sum_{i,j=1}^3 \left(\frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial q}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial q}{\partial x_j} \right) \right) dx \\ &= \int_{\partial K} \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} \left(n_j \frac{\partial q}{\partial x_i} - n_i \frac{\partial q}{\partial x_j} \right) d\sigma. \end{aligned}$$

Using the inequalities

$$\int_{\partial K} r^{-2\gamma+1} |\nabla q|^2 \, d\sigma \leq \|\nabla q\|_{V_{1-\gamma}^{1/2}(\partial K)}^2 \leq c \|q\|_{V_{1-\gamma}^2(K)}^2$$

and

$$\int_{\partial K} r^{2\gamma-1} \left| \frac{\partial u}{\partial x_j} \right|^2 \, d\sigma \leq \int_{\partial K} (r^{2\beta-1} + r^{2\beta}) \left| \frac{\partial u}{\partial x_j} \right|^2 \, d\sigma \leq c \left\| \frac{\partial u}{\partial x_j} \right\|_{E_\beta^1(K)}^2$$

(cf. (8) and (9)), we can easily show that the above identity holds also for $q \in V_{-\beta}^1(K) \cap V_{1-\gamma}^2(K)$. \square

We employ a regularity result for solutions of the problem (24). For this, we have to consider the following operator pencil $\mathcal{N}(\lambda)$ which is defined as

$$\mathcal{N}(\lambda) U = \left(-\delta U - \lambda(\lambda + n - 2)U, \frac{\partial U}{\partial \vec{n}}|_{\partial \Omega} \right) \quad \text{for } U \in W^2(\Omega).$$

As is known (see e.g. [9, Section 2.3]), the eigenvalues of this pencil are real and generalized eigenfunctions do not exist. The spectrum contains, in particular, the simple eigenvalues $\mu_1^+ = 0$ and $\mu_1^- = -1$ with the eigenfunction $\phi_1 = \text{const}$. The interval $(-1, 0)$ is free of eigenvalues. Let $\mu_2^+ \leq \mu_3^+ \leq \dots$ be the positive and $\mu_j^- = -1 - \mu_j^+$ the negative eigenvalues of the pencil \mathcal{N} .

Lemma 2.6 1) Suppose that $F \in (V_{-\beta}^1(K))^*$ and that $-\beta - 1/2$ is not an eigenvalue of the pencil $\mathcal{N}(\lambda)$. Then there exists a unique solution $p \in V_\beta^1(K)$ of the problem (24).

2) Suppose that $p \in V_\beta^1(K)$ is a solution of the problem (24), where $F \in (V_{-\beta}^1(K))^* \cap (V_{1-\gamma}^2(K))^*$, $\beta < \gamma$. We assume that the numbers $-\beta - 1/2$ and $-\gamma - 1/2$ are not eigenvalues of the pencil $\mathcal{N}(\lambda)$. Then

$$p = \sum c_j^\pm r^{\mu_j^\pm} \phi_j(\omega) + v$$

where μ_j^\pm are the eigenvalues of the pencil $\mathcal{N}(\lambda)$ in the interval $-\gamma - 1/2 < \lambda < -\beta - 1/2$, ϕ_j are corresponding eigenfunctions, $v \in V_{\gamma-1}^0(K)$ and

$$\|v\|_{V_{\gamma-1}^0(K)} \leq c \|F\|_{(V_{1-\gamma}^2(K))^*}$$

with a constant c independent of F .

P r o o f. The first assertion can be found e.g. in [15, Theorem 7.3.7]. We prove the second assertion. Under the conditions of the lemma, the operator

$$V_{1-\gamma}^2(K) \ni u \rightarrow \left(-\Delta u, \frac{\partial u}{\partial n} \right) \in V_{1-\gamma}^0(K) \times V_{1-\gamma}^{1/2}(\partial K)$$

is an isomorphism (see, e.g., [8, Theorem 6.1.1]). Consequently, the adjoint operator is also an isomorphism. This means that for arbitrary $F \in (V_{1-\gamma}^2(K))^*$ there exists a pair $(v, \phi) \in V_{\gamma-1}^0(K) \times V_{\gamma-1}^{-1/2}(\partial K)$ such that

$$-\int_K v \Delta q \, dx + \int_{\partial K} \phi \frac{\partial q}{\partial n} \, d\sigma = \langle F, q \rangle \quad \text{for all } q \in V_{1-\gamma}^2(K).$$

This together with (24) yields

$$-\int_K (v - p) \Delta q \, dx + \int_{\partial K} (\phi - p) \frac{\partial q}{\partial n} \, d\sigma = 0 \quad \text{for all } q \in V_{1-\beta}^2(K) \cap V_{1-\gamma}^2(K).$$

Thus, the triple $(v - p, \phi - p|_{\partial K}, 0)$ can be understood as a generalized solution of the Neumann problem

$$-\Delta(v - p) = 0 \text{ in } K, \quad \frac{\partial(v - p)}{\partial n} = 0 \text{ on } \partial K \setminus \{0\}$$

in the sense of [8, Section 3.2]. From [8, Lemma 3.2.4] we conclude that $\chi(v - p) \in W^2(K)$ for every $\chi \in C_0^\infty(\overline{K} \setminus \{0\})$. Let $\zeta = \zeta(r)$ be an infinitely differentiable function on $(0, \infty)$ such that $\zeta(r) = 1$ for $0 < r < 1$, $\zeta(r) = 0$ for $r > 2$, and let $\eta = 1 - \zeta$. With the notation $r = |x|$, the functions ζ and η can be also considered as functions on K . Obviously $\zeta(v - p) \in V_{\beta-1}^0(K)$ and $\eta(v - p) \in V_{\gamma-1}^0(K)$. Applying [8, Lemma 6.3.1], we obtain $\zeta(v - p) \in V_{\beta+1}^2(K)$ and $\eta(v - p) \in V_{\gamma+1}^2(K)$. Furthermore,

$$\Delta(\zeta p - \zeta v) = -\Delta(\eta p - \eta v) \in V_{\beta+1}^0(K) \cap V_{\gamma+1}^0(K), \quad \frac{\partial(\zeta p - \zeta v)}{\partial n} = \frac{\partial(\eta p - \eta v)}{\partial n} = 0 \text{ on } \partial K \setminus \{0\}.$$

Using [8, Theorem 6.1.4], we get

$$\zeta(p - v) = \sum c_j^\pm r^{\mu_j^\pm} \phi_j(\omega) + \eta(v - p)$$

This proves the lemma. \square

2.4 Normal solvability of the mapping A_β

Theorem 2.1 *Suppose that $\operatorname{Re} s \geq 0$, $|s| = 1$, that the line $\operatorname{Re} \lambda = -\beta + 1/2$ does not contain eigenvalues of the pencil $\mathcal{L}(\lambda)$, and $-\beta - 1/2$ is not an eigenvalue of the pencil $\mathcal{N}(\lambda)$. Then the range of the operator (18) is closed and the kernel has finite dimension.*

P r o o f. It suffices to show that every $(u, p) \in (E_\beta^2(K) \cap \mathring{E}_\beta^1(K)) \times V_\beta^1(K)$ satisfies the estimate

$$\|u\|_{E_\beta^2(K)}^2 + \|p\|_{V_\beta^1(K)}^2 \leq c \left(\|f\|_{E_\beta^0(K)}^2 + \|g\|_{X_\beta^1(K)}^2 + \int_{c_1 < |x| < c_2}^K (|u|^2 + |\nabla u|^2 + |p|^2) dx \right)$$

with certain positive c_1 and $c_2 > c_1$. Multiplying (20) by $2^{2\nu\beta}$ and summing up over all integer $\nu \geq N$, we get

$$\begin{aligned} \int_{|x| > 2^N}^K r^{2\beta} (|D^2 u|^2 + |u|^2 + |\nabla p|^2) dx &\leq c \int_{|x| > 2^{N-1}}^K r^{2\beta} (|f|^2 + |\nabla g|^2 + r^{-2} |g|^2) dx \\ &+ c \left(\int_{|x| > 2^{N-1}}^K r^{2\beta-2} (|\nabla u|^2 + r^{-2} |u|^2 + |p|^2) dx + \|\eta_N g\|_{(V_{-\beta}^1(K))^*}^2 \right), \end{aligned} \quad (25)$$

where $\eta_N = \zeta_N + \zeta_{N+1} + \dots$. Here, the constant c is independent of u, p and s .

Let ζ be a two times continuously differentiable function on \overline{K} , $\zeta(x) = 1$ for $|x| < \epsilon$ and $\zeta(x) = 0$ for $|x| > 2\epsilon$. Then

$$-\Delta(\zeta u_i) + \partial_{x_i}(\zeta p) = \zeta f_i - s \zeta u_i - 2\nabla \zeta \cdot \nabla u_i - u_i \Delta \zeta + p \partial_{x_i} \zeta =: F_i$$

for $i = 1, 2, 3$ and

$$-\nabla \cdot (\zeta u) = \zeta g - u \cdot \nabla \zeta.$$

Since the line $\operatorname{Re} \lambda = 2 - \beta - n/2$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$, there exists a constant c such that

$$\|\zeta u\|_{V_\beta^2(K)} + \|\zeta p\|_{V_\beta^1(K)} \leq c \left(\|F\|_{V_\beta^0(K)} + \|\zeta g - u \cdot \nabla \zeta\|_{V_\beta^1(K)} \right),$$

where c is independent of ζ . If ϵ is sufficiently small, then it follows that

$$\|\zeta u\|_{E_\beta^2(K)}^2 + \|\zeta p\|_{V_\beta^1(K)}^2 \leq c \left(\|\zeta f\|_{V_\beta^0(K)}^2 + \|\zeta g\|_{V_\beta^1(K)}^2 \right) + C(\epsilon) \int_{\epsilon < |x| < 2\epsilon}^K (|u|^2 + |\nabla u|^2 + |p|^2) dx.$$

Combining this with (25), we obtain

$$\begin{aligned} \|u\|_{E_\beta^2(K)}^2 + \|p\|_{V_\beta^1(K)}^2 &\leq c \left(\|f\|_{V_\beta^0(K)}^2 + \|g\|_{V_\beta^1(K)}^2 + \|g\|_{(V_{-\beta}^1(K))^*}^2 \right. \\ &\quad \left. + \int_{\substack{K \\ |x| > \epsilon}} r^{2\beta-2} \left(|\nabla u|^2 + r^{-2} |u|^2 + |p|^2 \right) dx \right). \end{aligned} \quad (26)$$

Obviously,

$$\int_{\substack{K \\ |x| > N}} r^{2\beta} \left(r^{-2} |\nabla u|^2 + r^{-4} |u|^2 \right) dx \leq \frac{c}{N^2} \|u\|_{E_\beta^2(K)}^2 \quad (27)$$

We estimate the integral of $r^{2\beta-2}|p|^2$. By (24), there is the decomposition $p = p_1 + p_2$, where p_j are the solutions of the problems

$$\int_K \nabla p_j \cdot \nabla q \, dx = \langle \Phi_j, q \rangle \quad \text{for all } q \in V_{-\beta}^1(K)$$

with the functionals

$$\langle \Phi_1, q \rangle = \int_K ((f - \nabla g) \cdot \nabla q - sgq) \, dx, \quad \langle \Phi_2, q \rangle = \int_K (\Delta u + \nabla g) \cdot \nabla q \, dx.$$

Obviously, $\Phi_1 \in (V_{-\beta}^1(K))^*$ and $\Phi_2 \in (V_{-\beta}^1(K))^*$,

$$\|\Phi_1\|_{(V_{-\beta}^1(K))^*} \leq \|f\|_{V_\beta^0(K)} + \|\nabla g\|_{V_\beta^0(K)} + \|g\|_{(V_{-\beta}^1(K))^*}.$$

Moreover, by Lemma 2.5, $\Phi_2 \in (V_{1-\gamma}^2(K))^*$ for $\beta \leq \gamma \leq \beta + 1/2$ and

$$\begin{aligned} |\langle \Phi_2, q \rangle|^2 &= \left| \int_K \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \left(n_j \frac{\partial q}{\partial x_i} - n_i \frac{\partial q}{\partial x_j} \right) d\sigma \right|^2 \\ &\leq c \int_K r^{2\gamma-1} \left| \frac{\partial u}{\partial x_j} \right|^2 d\sigma \int_K r^{1-2\gamma} |\nabla q|^2 d\sigma \leq c \|u\|_{E_\beta^2(K)}^2 \|q\|_{V_{1-\gamma}^2(K)}^2. \end{aligned}$$

For the last inequality, we used (8) and (9). By Lemma 2.6,

$$\|p_1\|_{V_\beta^1(K)}^2 \leq c_1 \left(\|f\|_{V_\beta^0(K)}^2 + \|\nabla g\|_{V_\beta^0(K)}^2 + \|g\|_{(V_{-\beta}^1(K))^*}^2 \right).$$

Let $\gamma \in (\beta, \beta + 1/2]$ be such that the interval $-\gamma - 1/2 \leq \lambda \leq -\beta - 1/2$ is free of eigenvalues of the pencil $\mathcal{N}(\lambda)$. Then, by the second part of Lemma 2.6, $p_2 \in V_{\gamma-1}^0(K)$ and

$$\|p_2\|_{V_{\gamma-1}^0(K)} \leq c \|\Phi_2\|_{(V_{1-\gamma}^2(K))^*} \leq c_2 \|u\|_{E_\beta^2(K)}.$$

Consequently,

$$\begin{aligned} \int_{\substack{K \\ |x| > N}} r^{2\beta-2} |p|^2 \, dx &\leq 2 \|p_1\|_{V_{\beta-1}^0(K)}^2 + \frac{2}{N^{2(\gamma-\beta)}} \|p_2\|_{V_{\gamma-1}^0(K)}^2 \\ &\leq 2c_1 \left(\|f\|_{V_\beta^0(K)}^2 + \|\nabla g\|_{V_\beta^0(K)}^2 + \|g\|_{(V_{-\beta}^1(K))^*}^2 \right) + \frac{c_2^2}{N^{2(\gamma-\beta)}} \|u\|_{E_\beta^2(K)}^2. \end{aligned}$$

This together with (26) implies

$$\begin{aligned} \|u\|_{E_\beta^2(K)}^2 + \|p\|_{V_\beta^1(K)}^2 &\leq c \left(\|f\|_{V_\beta^0(K)}^2 + \|g\|_{V_\beta^1(K)}^2 + \|g\|_{(V_{-\beta}^1(K))^*}^2 \right) \\ &\quad + C(\epsilon, N) \int_{\substack{K \\ \epsilon < |x| < N}} \left(|\nabla u|^2 + |u|^2 + |p|^2 \right) dx \end{aligned}$$

if N is sufficiently large. The proof is complete. \square

The following two lemmas show that the conditions of the last theorem are necessary.

Lemma 2.7 *The assertion of Theorem 2.1 is not true if the line $\operatorname{Re} \lambda = -\beta + 1/2$ contains an eigenvalue of the pencil $\mathcal{L}(\lambda)$.*

P r o o f. Let λ be an eigenvalue of the pencil \mathcal{L} , $\operatorname{Re} \lambda = -\beta + 1/2$, and let (u_0, p_0) be an eigenvector corresponding to this eigenvalue. Then the pair $(v, q) = (r^\lambda u_0(\omega), r^{\lambda-1} p_0(\omega))$ is a solution of the system

$$-\Delta v + \nabla q = 0, \quad \nabla \cdot v = 0 \quad \text{in } K$$

satisfying the condition $v = 0$ on $\partial K \setminus \{0\}$. We define by ζ_ε a smooth function on \mathbb{R}_+ such that $\zeta_\varepsilon(r) = 1$ for $\varepsilon < r < 1$, $\zeta_\varepsilon = 0$ outside the interval $(\varepsilon/2, 2)$ and $|\zeta_\varepsilon^{(j)}(r)| \leq c r^{-j}$ for $j \leq 2$, where the constant c is independent of r and ε . Obviously, the functions $u(x) = \zeta_\varepsilon(r) v(x)$ and $p(x) = \zeta_\varepsilon(r) q(x)$ satisfy the estimate

$$\|u\|_{E_\beta^2(K)}^2 + \|p\|_{V_\beta^1(K)}^2 \geq \int_K \left(r^{2\beta-4} |u|^2 + r^{2\beta-2} |p|^2 \right) dx \geq c \int_\varepsilon^1 r^{2\beta+2\operatorname{Re}\lambda-2} dr = -c \log \varepsilon.$$

Obviously, $|u| \leq c r^{\operatorname{Re}\lambda}$ and $|\nabla p - \Delta u| \leq c r^{\operatorname{Re}\lambda-2}$. Furthermore, $\nabla p - \Delta u$ vanishes outside the sets $\{x : \varepsilon/2 < |x| < \varepsilon\}$ and $\{x : 1 < |x| < 2\}$. Thus,

$$\begin{aligned} \|su - \Delta u + \nabla p\|_{E_\beta^0(K)}^2 &\leq c \left(\int_0^2 r^{2\beta+2\operatorname{Re}\lambda+2} dr + \int_{\varepsilon/2}^\varepsilon r^{2\beta+2\operatorname{Re}\lambda-2} dr + \int_1^2 r^{2\beta+2\operatorname{Re}\lambda-2} dr \right) \\ &= 2c(2 + \log 2). \end{aligned}$$

Using the estimates $|\nabla \cdot u| \leq c r^{\operatorname{Re}\lambda-1}$, $|\nabla(\nabla \cdot u)| \leq c r^{\operatorname{Re}\lambda-2}$ and the fact that $\nabla \cdot u = v \cdot \nabla \zeta_\varepsilon$ vanishes outside the sets $\{x : \varepsilon/2 < |x| < \varepsilon\}$ and $\{x : 1 < |x| < 2\}$, we analogously obtain

$$\|\nabla \cdot u\|_{V_\beta^1(K)} \leq C \quad \text{and} \quad \|\nabla \cdot u\|_{(V_\beta^1(K))^*} \leq \|\nabla \cdot u\|_{V_{\beta+1}^0(K)} \leq C.$$

The space $E_\beta^2(K)$ is compactly imbedded into $V_{\beta-1}^0(K)$, while $V_\beta^1(K)$ is compactly imbedded into the space $V_{\beta,\beta-2}^0(K)$ with the norm

$$\|p\|_{V_{\beta,\beta-2}^0(K)} = \left(\int_{\substack{K \\ |x|<1}} r^{2\beta} |p|^2 dx + \int_{\substack{K \\ |x|>1}} r^{2\beta-4} |p|^2 dx \right)^{1/2}.$$

One can easily check that the $V_{\beta-1}^0(K)$ -norm of u and the $V_{\beta,\beta-2}^0(K)$ of p have an upper bound C which is independent of ε . Thus, there does not exist a constant c independent of ε such that

$$\begin{aligned} \|u\|_{E_\beta^2(K)} + \|p\|_{V_\beta^1(K)} &\leq c \left(\|su - \Delta u + \nabla p\|_{V_\beta^0(K)} + \|\nabla \cdot u\|_{X_\beta^1(K)} \right. \\ &\quad \left. + \|u\|_{V_{\beta-1}^0(K)} + \|p\|_{V_{\beta,\beta-2}^0(K)} \right). \end{aligned} \quad (28)$$

The result follows. \square

We prove the same result concerning the condition on the eigenvalues of the pencil $\mathcal{N}(\lambda)$.

Lemma 2.8 *The assertion of Theorem 2.1 is not true if $-\beta - 1/2$ is an eigenvalue of the pencil $\mathcal{N}(\lambda)$.*

P r o o f. It suffices to show that there does not exist a constant c such that the estimate (28) is valid for all $u \in E_\beta^2(K) \cap \overset{\circ}{E}_\beta^1(K)$ and $p \in V_\beta^1(K)$ if $-\beta - 1/2$ is an eigenvalue of the pencil $\mathcal{N}(\lambda)$.

Let ϕ be an eigenfunction corresponding to the eigenvalue $\lambda = -\beta - 1/2$ of the pencil \mathcal{N} . Then the function $p_0(x) = r^\lambda \phi(\omega)$ is a solution of the problem

$$\Delta p_0 = 0 \quad \text{in } K, \quad \frac{\partial p_0}{\partial n} = 0 \quad \text{on } \partial K \setminus \{0\}.$$

Obviously, $\nabla p_0 = r^{\lambda-1} \psi(\omega)$ with a certain smooth function ψ on Ω . We define $v_0 = -s^{-1} \nabla p_0$. Then

$$(s - \Delta)v_0 + \nabla p_0 = 0, \quad \nabla \cdot v_0 = 0 \quad \text{in } K$$

and $v_0 \cdot n = 0$ on $\partial K \setminus \{0\}$. We correct the function v_0 by a term with support in a neighborhood of the boundary. Let $\nu(x)$ denote the distance of x from the boundary ∂K . Obviously the function ν is

positively homogeneous of degree 1. Furthermore, it is smooth in the neighborhood $\nu < \delta|x|$ of the boundary ∂K , where δ is a sufficiently small positive number. Moreover, $|\nabla \nu| = 1$ for $\nu < \delta|x|$ and $\nabla \nu(x)$ is orthogonal to ∂K for $x \in \partial K$.

We introduce the functions

$$v_{0,\nu} = v_0 \cdot \nabla \nu \quad \text{and} \quad v_{0,\tau} = v_0 - v_{0,\nu} \nabla \nu$$

in the neighborhood $\nu < \delta|x|$ of ∂K and define

$$u_0(x) = v_0(x) - \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} v_{0,\tau}(x),$$

where χ is a smooth cut-off function, $\chi = 1$ in $(0, \delta/2)$, $\chi = 0$ in (δ, ∞) . For $\nu > \delta|x|$, we set $u_0(x) = v_0(x)$. Here by \sqrt{s} , we mean the square root of s with positive real part. Then $u_0 = v_0 - v_{0,\tau} = v_{0,\nu} \nabla \nu = 0$ on ∂K . Since $(s - \Delta) e^{-\nu\sqrt{s}} = \sqrt{s} e^{-\nu\sqrt{s}} \Delta \nu$, we obtain

$$(s - \Delta)u_0 + \nabla p_0 = -(s - \Delta) \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} v_{0,\tau} = \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} f_0 + F_0, \quad (29)$$

where

$$f_0 = -\sqrt{s} v_{0,\tau} \Delta \nu - 2\sqrt{s} \sum_{j=1}^3 \frac{\partial \nu}{\partial x_j} \frac{\partial v_{0,\tau}}{\partial x_j} + \Delta v_{0,\tau}, \quad F_0 = \left[\Delta, \chi\left(\frac{\nu}{r}\right) \right] e^{-\nu\sqrt{s}} v_{0,\tau}$$

(here $[\Delta, \chi] = \Delta \chi - \chi \Delta$ denotes the commutator of Δ and χ). Furthermore,

$$\nabla \cdot u_0 = -\nabla \cdot \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} v_{0,\tau} = -\chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} \nabla \cdot v_{0,\tau} - e^{-\nu\sqrt{s}} v_{0,\tau} \cdot \nabla \chi\left(\frac{\nu}{r}\right) \quad (30)$$

since $v_{0,\tau} \cdot \nabla e^{-\nu\sqrt{s}} = -\sqrt{s} e^{-\nu\sqrt{s}} v_{0,\tau} \cdot \nabla \nu = 0$. Let ζ_N be a two times continuously differentiable function on $\mathbb{R}_+ = (0, \infty)$ such that

$$\zeta_N(r) = 0 \quad \text{for } r < 1 \text{ and } r > 2N, \quad \zeta_N(r) = 1 \quad \text{for } 2 < r < N.$$

We may assume that $|r^k \zeta_N^{(k)}(r)| \leq c$ for $k = 0, 1, 2$ with a constant c independent of N . We consider the functions

$$u = \zeta_N u_0 \quad \text{and} \quad p = \zeta_N p_0$$

Obviously, $u \in E_\beta^2(K)$, $p \in V_\beta^1(K)$ and $u = 0$ on ∂K . It can be easily checked that

$$\|u\|_{E_\beta^2(K)} + \|p\|_{V_\beta^1(K)} \geq \|\zeta_N(r) p_0\|_{V_{\beta-1}^0(K)} \geq C \log N$$

with a positive constant C . We estimate the $E_\beta^0(K)$ -norm of $(s - \Delta)u + \nabla p$. Obviously, $(s - \Delta)u + \nabla p = F_1 + F_2$, where

$$F_1 = \zeta_N((s - \Delta)u_0 + \nabla p_0), \quad \text{and} \quad F_2 = p_0 \nabla \zeta_N - 2 \sum_{j=1}^3 \frac{\partial \zeta_N}{\partial x_j} \frac{\partial u_0}{\partial x_j} - u_0 \Delta \zeta_N.$$

Here, $|F_1| \leq c r^{\lambda-2}$ and $|F_2| \leq c r^{\lambda-1}$. Since F_2 is zero outside the regions $1 < |x| < 2$ and $N < |x| < 2N$, we get

$$\|(s - \Delta)u + \nabla p\|_{E_\beta^0(K)}^2 \leq c \left(\int_1^{2N} r^{-3} dr + \int_1^2 r^{-1} dr + \int_N^{2N} r^{-1} dr \right) < \frac{c}{2} + 2c \log 2.$$

Next, we estimate the $X_\beta^1(K)$ -norm of $\nabla \cdot u$. Since $|\partial_x^\alpha \nabla \cdot u_0| \leq c r^{\lambda-2}$ for $|\alpha| \leq 1$ and $r > 1$, we have $|\partial_x^\alpha \nabla \cdot u| \leq c r^{\lambda-2}$ for $|\alpha| \leq 1$. Thus,

$$\|\nabla \cdot u\|_{V_\beta^1(K)}^2 \leq c \int_1^{2N} r^{-3} dr < \frac{c}{2}.$$

Since $|u_0| \leq c r^{\lambda-1}$ und $|\nabla \zeta_N| \leq c r^{-1}$, we get

$$\|u_0 \cdot \nabla \zeta_N\|_{(V_{-\beta}^1(K))^*}^2 \leq \|u_0 \cdot \nabla \zeta_N\|_{V_{\beta+1}^0(K)}^2 \leq c \left(\int_1^2 r^{-1} dr + \int_N^{2N} r^{-1} dr \right) = 2c \log 2.$$

Furthermore, we conclude from (30) that $|\zeta_N \nabla \cdot u_0| \leq c r^{\lambda-2} e^{-\nu \sqrt{s}}$. Consequently,

$$\|\zeta_N \nabla \cdot u_0\|_{(V_{-\beta}^1(K))^*}^2 \leq \|\zeta_N \nabla \cdot u_0\|_{V_{\beta+1}^0(K)}^2 \leq c \int r^{-3} e^{-2\nu(x) \operatorname{Re} \sqrt{s}} dx,$$

where the integration is extended over the set of all $x \in K$ such that $1 < r = |x| < 2N$ and $\nu(x) < \delta r$. For arbitrary $x \in K$, $\nu(x) < \delta|x|$, let $\tau(x)$ be the nearest point to x on ∂K . Obviously, $|\tau(x)| \leq |x| \leq (1-\delta)^{-1/2}|\tau(x)|$. This implies

$$\|\zeta_N \nabla \cdot u_0\|_{(V_{-\beta}^1(K))^*}^2 \leq c \int_{\substack{\partial K \\ |\tau|^2 > 1-\delta}}^{\delta r} \int_0^{\delta r} |\tau|^{-3} e^{-2\nu \operatorname{Re} \sqrt{s}} d\nu d\tau \leq \frac{c}{2 \operatorname{Re} \sqrt{s}} \int_{\substack{\partial K \\ |\tau|^2 > 1-\delta}} |\tau|^{-3} d\tau \leq \frac{c'}{\operatorname{Re} \sqrt{s}}.$$

The last estimates yield

$$\|\nabla \cdot u\|_{X_\beta^1(K)} \leq C$$

with a constant C independent of N . Finally, since $|p(x)| \leq c r^\lambda$, $|u(x)| \leq c r^{\lambda-1}$ and $u(x), p(x)$ are zero for $|x| < 1$, we get the estimate

$$\|u\|_{V_{\beta-1}^0(K)}^2 + \|p\|_{V_{\beta,\beta-2}^0(K)}^2 \leq c \int_{\substack{K \\ |x| > 1}} r^{2\beta+2\lambda-4} dx = C < \infty.$$

Thus, the right-hand side of (28) has an upper bound independent of N for our pair (u, p) , while the left-hand side can be estimate from below by $C \log N$ with positive C . This proves the lemma. \square

2.5 A regularity result for solutions of the problem (7)

Using Lemma 2.6 and regularity results for solutions of the stationary Stokes system, we can prove the following lemma.

Lemma 2.9 *Suppose that $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ is a solution of the problem (7), where*

$$f \in E_\beta^0(K) \cap E_\gamma^0(K), \quad g \in X_\beta^1(K) \cap X_\gamma^1(K).$$

We assume that one of the following two conditions is satisfied:

- (i) $\beta < \gamma$ and the interval $-\gamma - 1/2 \leq \lambda \leq -\beta - 1/2$ does not contain eigenvalues of the pencil $\mathcal{N}(\lambda)$,
- (ii) $\beta > \gamma$ and the strip $-\beta + 1/2 \leq \operatorname{Re} \lambda \leq -\gamma + 1/2$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$.

Then $u \in E_\gamma^2(K)$, $p \in V_\gamma^1(K)$ and

$$\|u\|_{E_\gamma^2(K)} + \|p\|_{V_\gamma^1(K)} \leq c \left(\|f\|_{E_\gamma^0(K)} + \|g\|_{X_\gamma^1(K)} + \|u\|_{E_\beta^2(K)} + \|p\|_{V_\beta^1(K)} \right).$$

P r o o f. 1) Suppose that the condition (i) is satisfied. In addition, let $\gamma \leq \beta + 1/2$. Then p is a solution of the problem (24), where $F \in (V_{-\beta}^1(K))^* \cap (V_{1-\gamma}^2(K))^*$,

$$\langle F, q \rangle = \int_K ((f + \Delta u) \cdot \nabla q - s g q) dx \quad \text{for } q \in V_{-\beta}^1(K) \quad \text{and} \quad (31)$$

$$\langle F, q \rangle = \int_K ((f - \nabla g) \cdot \nabla q - s g q) dx + \int_{\partial K} \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \left(n_j \frac{\partial q}{\partial x_i} - n_i \frac{\partial q}{\partial x_j} \right) d\sigma \quad (32)$$

for $q \in V_{1-\gamma}^2(K)$ (cf. Lemma 2.5). From Lemma 2.6 we conclude that $p \in V_{\gamma-1}^0(K)$. Furthermore, $u \in E_\beta^2(K) \subset V_{\beta-1}^1(K) \cap V_\beta^1(K) \subset V_{\gamma-1}^1(K)$. Applying Lemma 2.4, we obtain $u \in E_\gamma^2(K)$ and $p \in V_\gamma^1(K)$. This proves the lemma for the case $\gamma \leq \beta + 1/2$. By induction, the assertion of the lemma holds for arbitrary $\gamma > \beta$ provided that the interval $[-\gamma - 1/2, -\beta - 1/2]$ is free of eigenvalues of the pencil $\mathcal{N}(\lambda)$.

2) We assume now that the condition (ii) is satisfied. Let ζ be a smooth function on \overline{K} with compact support which is equal to one near the vertex of the cone. Furthermore, let $\eta = 1 - \zeta$.

Obviously, $\eta(u, p) \in E_\gamma^2(K) \times V_\gamma^2(K)$ since $\gamma < \beta$. Furthermore, the pair $(\zeta u, \zeta p)$ satisfies the equations

$$-\Delta(\zeta u) + \nabla(\zeta p) = F, \quad -\nabla \cdot (\zeta u) = G \quad \text{in } K$$

and the boundary condition $\zeta u = 0$ on $\partial K \setminus \{0\}$, where

$$F = \zeta f - s\zeta u - 2 \sum_{j=1}^3 \frac{\partial \zeta}{\partial x_j} \frac{\partial u}{\partial x_j} - u \Delta \zeta + p \nabla \zeta, \quad G = \zeta g - u \cdot \nabla \zeta.$$

Suppose first that $\beta - 2 \leq \gamma < \beta$. Then $F \in V_\gamma^0(K)$ and $G \in V_\gamma^1(K)$. Since the strip $-\beta + 1/2 \leq \operatorname{Re} \lambda \leq -\gamma + 1/2$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$ it follows that $\zeta(u, p) \in V_\gamma^2(K) \times V_\gamma^1(K)$ (see, e. g., [17, Chapter 3, Theorem 5.6]). By induction, the same result holds for $\gamma < \beta - 2$. Thus, $(u, p) \in E_\gamma^2(K) \times V_\gamma^1(K)$. \square

Note that the result of Lemma 2.9 is also true if $\beta < \gamma$, the interval $-\gamma - 1/2 \leq \lambda \leq -\beta - 1/2$ contains only the eigenvalue $\mu_1^- = -1$ as an inner point, and $\int_K g \, dx = 0$ (see Lemma 2.13 below).

2.6 Injectivity of the operator A_β

Let λ_1^+ be the smallest positive eigenvalue of the pencil $\mathcal{L}(\lambda)$ and let $\lambda_1^- = -1 - \lambda_1^+$. Then the strip $\lambda_1^- < \operatorname{Re} \lambda < \lambda_1^+$ is the widest strip containing the line $\operatorname{Re} \lambda = -1/2$ which is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$. As mentioned above, $\lambda_1^+ \leq 1$ and $\lambda_1^- \geq -2$.

Lemma 2.10 *Suppose that $\operatorname{Re} s \geq 0$, $|s| = 1$, $-\mu_2^+ - 1/2 < \beta < \lambda_1^+ + 3/2$ and $\beta \neq -1/2$. Then A_β is injective.*

P r o o f. 1) If $0 \leq \beta \leq 1$, then the assertion of the lemma follows from Theorem 1.1 and Lemma 2.1.

2) Suppose that $1 < \beta < \lambda_1^+ + 3/2$, i.e., $\lambda_1^- < -\beta + 1/2 < -1/2$. Then the strip $-\beta + 1/2 \leq \operatorname{Re} \lambda \leq -1/2$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$ and it follows from Lemma 2.9 that $\ker A_\beta \subset \ker A_1$. Since A_1 is injective, it follows that the kernel of A_β is trivial.

3) If $-1/2 < \beta < 0$, then the interval $-1/2 \leq \lambda \leq -\beta - 1/2$ is free of eigenvalues of the pencil $\mathcal{N}(\lambda)$ (even the interval $(-1, 0)$ is free of eigenvalues of this pencil). Thus, it follows from Lemma 2.9 that $\ker A_\beta \subset \ker A_0$. This implies that $\ker A_\beta$ is trivial.

4) We consider the case $\max(-1, -\mu_2^+ - 1/2) < \beta < -1/2$ and set $\gamma = \beta + 1/2$. If $(u, p) \in \ker A_\beta$, then

$$\int_K \nabla p \cdot \nabla q \, dx = \langle F, q \rangle = \int_K \Delta u \cdot \nabla q \, dx \quad \text{for all } q \in V_{-\beta}^1(K).$$

By Lemma 2.5, the functional F is continuous on $V_{-\beta}^1(K)$ and on $V_{1-\gamma}^2(K)$. Since $\mu_1^- = -1 < -\gamma - 1/2 < 0 < -\beta - 1/2 < \mu_2^+$, the interval $-\gamma - 1/2 \leq \lambda \leq -\beta - 1/2$ contains only the eigenvalue $\mu_1^+ = 0$ of the pencil $\mathcal{N}(\lambda)$, and the corresponding eigenfunction is constant. Thus, we get the decomposition $p = c + q$, where $q \in V_{\gamma-1}^1$ and c is a constant (see Lemma 2.6). Obviously, $E_\beta^2(K) \subset V_{\gamma-1}^1(K)$. This means that the pair (u, q) belongs to $V_{\gamma-1}^1(K) \times V_{\gamma-1}^0(K)$. Furthermore,

$$su - \Delta u + \nabla q = 0, \quad \nabla \cdot u = 0 \quad \text{in } K, \quad u = 0 \quad \text{on } \partial K \setminus \{0\}.$$

Using Lemma 2.4, we conclude that $(u, q) \in E_\gamma^2(K) \times V_\gamma^1(K)$, i.e., $(u, q) \in \ker A_\gamma$. As was shown above, the kernel of A_γ is trivial. This means that $u = 0$ and $q = 0$. The last implies that p is constant in K . However, the space $V_\beta^1(K)$ contains only the constant $p = 0$. This shows that $\ker A_\beta$ is trivial.

5) Suppose that $\mu_2^+ > 1/2$ and $-\mu_2^+ - 1/2 < \beta \leq -1$. Let γ be an arbitrary number in the interval $(-1, -\frac{1}{2})$. Then $\mu_1^+ = 0 < -\gamma - 1/2 < -\beta - 1/2 < \mu_2^+$. Since the interval $-\gamma - 1/2 \leq \lambda \leq -\beta - 1/2$ is free of eigenvalues of the pencil $\mathcal{N}(\lambda)$, it follows from Lemma 2.9 that $\ker A_\beta \subset \ker A_\gamma$. As was shown above, the kernel of A_γ and, therefore, also the kernel of A_β are trivial. The proof is complete. \square

2.7 Bijectivity of the operator of the problem (7)

We are interested in conditions on β which ensure the bijectivity of the operator A_β . By Lemma 2.8, the operator A_β is not Fredholm if $-\beta - 1/2$ (and, therefore, also $\beta - 1/2$) is an eigenvalue of the pencil $\mathcal{N}(\lambda)$. In particular, we must exclude the value $\beta = 1/2 = \mu_1^+ + 1/2$. We consider the cases $\beta < 1/2$ and $\beta > 1/2$ separately.

1) The case $\beta < 1/2$

In order to show that the operator A_β is surjective for $-\lambda_1^+ + 1/2 < \beta < 1/2$, we prove the following regularity assertion for weak solutions of the problem (7).

Lemma 2.11 *Let $|s| = 1$, $\operatorname{Re} s \geq 0$, and let $(u, p) \in \mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K))$ be the solution of the problem (12), where*

$$f \in E_0^{-1}(K) \cap E_\beta^0(K), \quad g \in L_2(K) \cap (V_0^1(K))^* \cap X_\beta^1(K)$$

and $-\lambda_1^+ + 1/2 < \beta < 1/2$. Then $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$.

P r o o f. Let ζ be the same cut-off function as in the proof of Lemma 2.9, and let $\eta = 1 - \zeta$. Obviously, $\zeta(u, p) \in \mathring{V}_0^1(K) \times L_2(K)$,

$$-\Delta(\zeta u) + \nabla(\zeta p) = \zeta f - \zeta u + p \nabla \zeta + \zeta \Delta u - \Delta(\zeta u) \in V_\beta^0(K)$$

and $-\nabla \cdot (\zeta u) = \zeta g - u \cdot \nabla \zeta \in V_\beta^1(K)$. By our assumption on β , the strip $-1/2 \leq \operatorname{Re} \lambda \leq -\beta + 1/2$ does not contain eigenvalues of the pencil $\mathcal{L}(\lambda)$. Using regularity results for solutions of the Stokes system (see, e.g., [17, Chapter 3, Theorem 5.6] and [15, Theorems 10.3.1 and 10.3.4]), we conclude that $\zeta u \in V_\beta^2(K)$ and $\zeta p \in V_\beta^1(K)$. Since ζ has compact support, this implies $\zeta(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$.

It remains to prove that $\eta(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$. Obviously,

$$s\eta u - \Delta(\eta u) + \nabla(\eta p) = F, \quad -\nabla \cdot (\eta u) = G, \quad (33)$$

where $F = \eta f + p \nabla \eta + \eta \Delta u - \Delta(\eta u) \in V_\beta^0(K)$ and $G = \eta g - u \cdot \nabla \eta \in X_\beta^1(K)$. Suppose first that $\frac{1}{2} - \lambda_1^+ < \beta < 0$ (this requires that $\lambda_1^+ > 1/2$). Then it follows from our assumptions that $\eta u \in V_{\beta-1}^1(K)$ and $\eta p \in V_{\beta-1}^0(K)$. Applying Lemma 2.4, we get $\eta(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$.

Suppose now that $\max(0, -\lambda_1^+ + 1/2) < \beta < 1/2$. Since $\beta > 0$ and F and G are zero near the vertex of the cone, we have $F \in E_0^0(K)$ and $G \in X_0^1(K)$. Furthermore, $\eta u \in V_{-1}^1(K)$ and $\eta p \in V_{-1}^0(K)$. Thus, Lemma 2.4 implies $\eta(u, p) \in E_0^2(K) \times V_0^1(K)$. On the other hand, $F \in E_\beta^0(K)$ and $G \in X_\beta^1(K)$. Since the interval $-\beta - 1/2 \leq \lambda \leq -1/2$ is free of eigenvalues of the pencil $\mathcal{N}(\lambda)$, we conclude from Lemma 2.9 that $\eta(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$. The proof is complete. \square

As a consequence of the last two lemmas and Theorem 2.1, we obtain the following result.

Theorem 2.2 *Suppose that $|s| = 1$, $\operatorname{Re} s \geq 0$ and $\frac{1}{2} - \lambda_1^+ < \beta < \frac{1}{2}$. Then the operator (18) is an isomorphism.*

P r o o f. Since $0 < \frac{1}{2} - \beta < \lambda_1^+$ and $-1 < -\frac{1}{2} - \beta < \lambda_1^+ - 1 \leq 0$, the line $\operatorname{Re} \lambda = \frac{1}{2} - \beta$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$ and the line $\operatorname{Re} \lambda = -\frac{1}{2} - \beta$ is free of eigenvalues of the pencil $\mathcal{N}(\lambda)$. Thus by Theorem 2.1, the operator (18) has closed range. Furthermore by Lemma 2.10, the kernel of this operator is trivial. From Theorem 1.1 and Lemma 2.11 we conclude that the range of the operator (18) contains the set

$$(E_0^{-1}(K) \cap E_\beta^0(K)) \times (L_2(K) \cap (V_0^1(K))^* \cap X_\beta^1(K))$$

Since the range is closed it follows that the operator is an isomorphism onto $E_\beta^0(K) \times X_\beta^1(K)$. \square

1) The case $\beta > 1/2$

We consider the adjoint operator

$$E_{-\beta}^0(K) \times (X_\beta^1(K))^* \ni (v, q) \rightarrow A_\beta^*(v, q) \in (E_\beta^2(K) \cap \mathring{E}_\beta^1(K))^* \cap (V_\beta^1(K))^*$$

of A_β , where

$$\langle A_\beta^*(v, q), (u, p) \rangle = \int_K (v(su - \Delta u + \nabla p) - q \nabla \cdot u) dx$$

for all $u \in E_\beta^2(K) \cap \mathring{E}_\beta^1(K)$, $p \in V_\beta^1(K)$. We show that the constant vector-function $(v, q) = (0, 1)$ is an element of the kernel of A_β^* if $1/2 < \beta < 5/2$.

Lemma 2.12 *The constant function $q = 1$ belongs to $(X_\beta^1(K))^*$ if and only if $1/2 < \beta < 5/2$. In this case, the pair $(v, q) = (0, 1)$ is an element of $\ker A_\beta^*$, and A_β is a mapping from $(E_\beta^2(K) \cap \mathring{E}_\beta^1(K)) \times V_\beta^1(K)$ into the space*

$$\left\{ (f, g) \in E_\beta^0(K) \times X_\beta^1(K) : \int_K g(x) dx = 0 \right\}. \quad (34)$$

P r o o f. Let $\zeta = \zeta(r)$ be a continuously differentiable function, $\zeta(r) = 1$ for $r < 1/4$, $\zeta(r) = 0$ for $r > 1/2$. In the case $1/2 < \beta < 5/2$, the $V_{1-\beta}^0(K)$ -norm of ζ and the $V_{-\beta}^1(K)$ -norm of $\eta = 1 - \zeta$ are finite, and we obtain

$$\left| \int_K g dx \right| \leq \left| \int_K \zeta g dx \right| + \left| \int_K \eta g dx \right| \leq \|\zeta\|_{V_{1-\beta}^0(K)} \|g\|_{V_\beta^1(K)} + \|\eta\|_{V_{-\beta}^1(K)} \|g\|_{(V_{-\beta}^1(K))^*}$$

for arbitrary $g \in X_\beta^1(K)$. In particular, this means that the constant function $q = 1$ belongs to the dual space of $X_\beta^1(K)$ if $1/2 < \beta < 5/2$. Therefore, we get

$$\int_K g dx = - \int_K q \nabla \cdot u dx = \int_K u \nabla q dx = 0$$

for $q = 1$, $u \in E_\beta^2(K) \cap \mathring{E}_\beta^1(K)$ and $g = -\nabla \cdot u$. In particular, $A_\beta^*(0, 1) = 0$.

We show that $1 \notin (X_\beta^1(K))^*$ if $\beta \leq 1/2$ or $\beta \geq 5/2$. Suppose that $\beta \geq 5/2$. We consider the function $g(x) = \zeta(r) r^{-3} (\log r)^{-1}$. One easily checks that $g \in V_\beta^1(K) \cap V_{\beta+1}^0(K) \subset X_\beta^1(K)$. However,

$$\left| \int_K g(x) dx \right| \geq c \left| \int_0^{1/4} \frac{dr}{r \log r} \right| = \infty.$$

In the case $\beta \leq 1/2$, the function $g(x) = \chi(r) r^{-3} (\log r)^{-1}$, where $\chi(r) = 0$ for $r < 2$ and $\chi(r) = 1$ for $r > 4$, is an element of the space $V_\beta^1(K) \cap V_{\beta+1}^0(K)$, whereas the integral of g over K is not finite. This shows that in both cases $1 \notin (X_\beta^1(K))^*$. The proof of the lemma is complete. \square

In order to prove the existence of solutions of the problem (7) in the space $E_\beta^2(K) \times V_\beta^1(K)$ for $\beta > 1/2$, we will apply the following regularity assertion.

Lemma 2.13 *Suppose that $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ is a solution of the problem (7), where*

$$f \in E_\beta^0(K) \cap E_\gamma^0(K), \quad g \in X_\beta^1(K) \cap X_\gamma^1(K),$$

$-1/2 < \beta < \gamma < \mu_2^+ + 1/2$, $\beta \neq 1/2$ and $\gamma \neq 1/2$. In the case $\beta < 1/2 < \gamma$, we assume in addition that $\int_K g(x) dx = 0$. Then $u \in E_\gamma^2(K)$ and $p \in V_\gamma^1(K)$.

P r o o f. In the cases $-1/2 < \beta < \gamma < 1/2$ and $1/2 < \beta < \gamma < \mu_2^+ + 1/2$, the interval $-\gamma - 1/2 \leq \lambda \leq -\beta - 1/2$ does not contain eigenvalues of the pencil $\mathcal{N}(\lambda)$, and the assertion follows from Lemma 2.9. Therefore, we may assume that $-1/2 < \beta < 1/2 < \gamma < \mu_2^+ + 1/2$. In this case, the number $\lambda = -1$ is the only eigenvalue of the pencil $\mathcal{N}(\lambda)$ in the interval $-\gamma - 1/2 \leq \lambda \leq -\beta - 1/2$. Suppose first that $\gamma \leq \beta + 1/2$. Then p is a solution of the problem (24), where $F \in (V_{-\beta}^1(K))^* \cap (V_{1-\gamma}^2(K))^*$. More precisely, F has the representation (31) on $V_{-\beta}^1(K)$ and the representation (32) on $V_{1-\gamma}^2(K)$. As was shown in the proof of Lemma 2.6, there exists a pair $(v, \phi) \in V_{\gamma-1}^0(K) \times V_{\gamma-1}^{-1/2}(\partial K)$ such that

$$- \int_K v \Delta q dx + \int_{\partial K} \phi \frac{\partial q}{\partial n} d\sigma = \langle F, q \rangle \quad \text{for all } q \in V_{1-\gamma}^2(K). \quad (35)$$

By Lemma 2.6, the function p admits the decomposition

$$p = c r^{-1} + v.$$

Let ζ be the same cut-off function as in the proof of Lemma 2.6, and let $\eta = 1 - \zeta$. Obviously, $\zeta \in V_{1-\beta}^2(K) \subset V_{-\beta}^1(K)$ and $\eta \in V_{1-\gamma}^2(K)$. Furthermore, $\frac{\partial \zeta}{\partial n} = \frac{\partial \eta}{\partial n} = 0$ on $\partial K \setminus \{0\}$. By (24) and Lemma 2.5, we have

$$-\int_K p \Delta \zeta \, dx = \langle F, \zeta \rangle = \int_K (f - su - \nabla g) \cdot \nabla \zeta \, dx + \int_{\partial K} \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \left(n_j \frac{\partial \zeta}{\partial x_i} - n_i \frac{\partial \zeta}{\partial x_j} \right) d\sigma,$$

Furthermore, (35) yields

$$\begin{aligned} \int_K (p - cr^{-1}) \Delta \zeta \, dx &= - \int_K v \Delta \eta \, dx = \langle F, \eta \rangle \\ &= \int_K ((f - \nabla g) \cdot \nabla \eta - sg \eta) \, dx + \int_{\partial K} \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \left(n_j \frac{\partial \eta}{\partial x_i} - n_i \frac{\partial \eta}{\partial x_j} \right) d\sigma. \end{aligned}$$

Adding the last two equalities, we obtain

$$-c \int_K r^{-1} \Delta \zeta \, dx = -s \int_K (\eta g + u \nabla \zeta) \, dx.$$

Since $u \in \mathring{E}_\beta^1(K)$, $\nabla \cdot u = -g \in (V_{-\beta}^1(K))^*$ and $\zeta \in V_{-\beta}^1(K)$, this implies

$$c \int_K r^{-1} \Delta \zeta \, dx = s \int_K g \, dx = 0.$$

The left-hand side of the last equality is equal to

$$c \operatorname{mes} \Omega \int_0^\infty (r \zeta''(r) + 2\zeta'(r)) \, dr = -c \operatorname{mes} \Omega,$$

Hence, we get $c = 0$ and $p = v \in V_{\gamma-1}^0(K)$. Furthermore, $u \in E_\beta^2(K) \subset V_{\gamma-1}^1(K)$. Applying Lemma 2.4, we obtain $u \in E_\gamma^2(K)$ and $p \in V_\gamma^1(K)$. This proves the lemma for the case $\gamma \leq \beta + 1/2$. By induction, the assertion of the lemma holds for arbitrary $\gamma \in (1/2, \mu_2^+ + 1/2)$. \square

Now, the following result holds analogously to Theorem 2.2.

Theorem 2.3 *Suppose that $|s| = 1$, $\operatorname{Re} s \geq 0$, $\frac{1}{2} < \beta < \min(\mu_2^+ + \frac{1}{2}, \lambda_1^+ + \frac{3}{2})$. Then the operator $(u, p) \rightarrow (f, g)$ of the problem (7) is an isomorphism from $(E_\beta^2(K) \cap \mathring{E}_\beta^1(K)) \times V_\beta^1(K)$ onto the space (34).*

P r o o f. By Theorem 2.1 and Lemma 2.10, the operator (18) has closed range and trivial kernel. Furthermore, the problem (7) is solvable in $E_\beta^2(K) \times V_\beta^1(K)$ for arbitrary $f \in C_0^\infty(\overline{K} \setminus \{0\})$ and $g \in C_0^\infty(\overline{K} \setminus \{0\})$ such that $\int_K g \, dx = 0$. Indeed, for such f and g , there exists a unique variational solution $(u, p) \in \mathring{E}_0^1(K) \times (L_2(K) + V_0^1(K))$ of the problem (7). Using Lemmas 2.11 and 2.13, we conclude that $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$. Since the set $C_0^\infty(\overline{K} \setminus \{0\})$ is dense in $E_\beta^0(K)$ and $X_\beta^1(K)$, it follows that the problem (7) is solvable for arbitrary $f \in E_\beta^0(K)$, $g \in X_\beta^1(K)$, $\int_K g \, dx = 0$. \square

2.8 Necessity of the conditions on β

We already proved (see Lemmas 2.7 and 2.8) that the operator A_β is not Fredholm if the line $\operatorname{Re} \lambda = -\beta + 1/2$ contains eigenvalues of the pencil $\mathcal{L}(\lambda)$ or if $-\beta - 1/2$ is an eigenvalue of the pencil $\mathcal{N}(\lambda)$. Our goal is to show that the conditions $\beta > -\lambda_1^+ + 1/2$ and $\beta < \min(\mu_2^+ + \frac{1}{2}, \lambda_1^+ + \frac{3}{2})$ in Theorems 2.2 and 2.3 are necessary.

Lemma 2.14 *Suppose that $-1/2 < \beta < -\lambda_1^+ + 1/2$. Then the kernel of the adjoint operator A_β^* is not trivial.*

P r o o f. We assume that $-1/2 < \beta < -\lambda_1^+ + 1/2$ and $\ker A_\beta^*$ is trivial. Then it follows from Theorem 2.1 and Lemma 2.10 that A_β is an isomorphism. Let (ϕ, ψ) be an eigenvector of the pencil $\mathcal{L}(\lambda)$ corresponding to the eigenvalue λ_1^+ and let ζ be a smooth ($\in C^2$) function with

compact support equal to one near the vertex of the cone. Furthermore, let δ be a real number, $-\lambda_1^+ + 1/2 < \delta < \min(\beta + 1, 1/2)$. We consider the functions

$$u(x) = \zeta(x) r^{\lambda_1^+} \phi(\omega), \quad p(x) = \zeta(x) r^{\lambda_1^+ - 1} \psi(\omega).$$

Obviously, $u \in E_\delta^2(K)$, $p \in V_\delta^1(K)$ and $u = 0$ on ∂K . Since

$$-\Delta r^{\lambda_1^+} \phi + \nabla r^{\lambda_1^+ - 1} \psi = 0, \quad -\nabla \cdot r^{\lambda_1^+} \phi = 0 \quad \text{in } K,$$

we get $su - \Delta u + \nabla p = f \in E_\beta^0(K) \cap E_\delta^0(K)$ and $-\nabla \cdot u = g \in X_\beta^1(K) \cap X_\delta^1(K)$. Since A_β is an isomorphism, there exists a pair $(v, q) \in E_\beta^2(K) \times V_\beta^1(K)$ such that $A_\beta(v, q) = (f, g)$. From Lemma 2.13 it follows that $(v, q) \in E_\delta^2(K) \times V_\delta^1(K)$. Therefore, $(u - v, p - q) \in \ker A_\delta$. Using Theorem 2.1, we conclude that $(u, p) = (v, q)$, i. e., $u \in E_\beta^2(K)$ and $p \in V_\beta^1(K)$. Since obviously $(u, p) \notin E_\beta^2(K) \times V_\beta^1(K)$, we got a contradiction. \square

Note that, in contrast to the case $1/2 < \beta < 5/2$, the kernel of A_β^* does not contain the constant vector $(0, 1)$ if β satisfies the condition of Lemma 2.14.

Next, we consider the case that $\beta > \min(\mu_2^+ + 1/2, \lambda_1^+ + 3/2)$.

Lemma 2.15 *Suppose that $\lambda_1^+ < \mu_2^+ - 1$. Then*

- 1) *the kernel of the operator A_β is not trivial for $\lambda_1^+ + 3/2 < \beta < \mu_2^+ + 1/2$.*
- 2) *the kernel of the operator A_β^* is not trivial for $-\mu_2^+ - 1/2 < \beta < -\lambda_1^+ + 1/2$.*

P r o o f. 1) Let $(\phi(\omega), \psi(\omega))$ be an eigenvector to the eigenvalue $\lambda_1^- = -1 - \lambda_1^+$ of the pencil $\mathcal{L}(\lambda)$. Furthermore, let $\zeta = \zeta(r)$ be a smooth function on $(0, \infty)$ with compact support which is equal to one near the point $r = 0$. We define $u_0(x) = \zeta(r) r^{\lambda_1^-} \phi(\omega)$ and $p_0(x) = \zeta(r) r^{\lambda_1^- - 1} \psi(\omega)$. One can easily check that $u_0 \in E_\beta^2(K)$ and $p_0 \in V_\beta^1(K)$ for $\beta > \lambda_1^+ + 3/2$. Since

$$-\Delta r^{\lambda_1^-} \phi(\omega) + \nabla r^{\lambda_1^- - 1} \psi(\omega) = 0 \quad \text{and} \quad -\nabla \cdot r^{\lambda_1^-} \phi(\omega) = 0 \quad \text{in } K$$

we get $|su_0 - \Delta u_0 + \nabla p_0| \leq c r^{\lambda_1^-}$. Furthermore, $\nabla \cdot u_0(x) = 0$ for small and for large $|x|$. Consequently, $su_0 - \Delta u_0 + \nabla p_0 \in E_\gamma^0(K)$ and $\nabla \cdot u_0 \in X_\gamma^1(K)$ for arbitrary $\gamma \in (\frac{1}{2}, \lambda_1^+ + \frac{3}{2})$. By Theorem 2.3, there exists a vector function $(u_1, p_1) \in E_\gamma^2(K) \times V_\gamma^1(K)$ such that

$$su_1 - \Delta u_1 + \nabla p_1 = su_0 - \Delta u_0 + \nabla p_0, \quad \nabla \cdot u_1 = \nabla \cdot u_0 \quad \text{in } K$$

and $u_1 = u_0 = 0$ on ∂K . Using Lemma 2.9, we conclude that $(u_1, p_1) \in E_\beta^2(K) \times V_\beta^1(K)$. This means that $(u, p) = (u_0 - u_1, p_0 - p_1)$ is an element of the kernel of A_β . Since $(u_0, p_0) \notin E_\gamma^2(K) \times V_\gamma^1(K)$, the pair (u, p) is not zero. This proves the first assertion.

2) We show that

$$\ker A_\beta^* \supset \ker A_{-\gamma} \quad \text{if } \gamma \leq \beta \leq \gamma + 2. \quad (36)$$

Let $(u, p) \in (E_{-\gamma}^2(K) \cap \mathring{E}_{-\gamma}^1(K)) \times V_{-\gamma}^1(K)$ be an element of the kernel of $A_{-\gamma}$. Then

$$\int_K ((su - \Delta u + \nabla p) \cdot v - q \nabla \cdot u) dx = 0 \quad \text{for all } v \in E_\gamma^0(K), \quad q \in (X_{-\gamma}^1(K))^*. \quad (37)$$

If $\gamma \leq \beta \leq \gamma + 2$, then

$$E_\beta^2(K) \subset E_\gamma^0(K) \quad \text{and} \quad E_{-\gamma}^2(K) \subset E_{-\beta}^0(K).$$

Furthermore,

$$V_\beta^1(K) \subset V_\gamma^1(K) + V_{\gamma+2}^1(K) \subset V_\gamma^1(K) + V_{\gamma+1}^0(K) \subset V_\gamma^1(K) + (V_{-\gamma}^1(K))^* = (X_{-\gamma}^1(K))^*$$

and $V_{-\gamma}^1(K) \subset (X_\beta^1(K))^*$. Thus, $(u, p) \in E_{-\beta}^0(K) \times (X_\beta^1(K))^*$ and the equality (37) is valid for all $v \in E_\beta^2(K) \cap \mathring{E}_\beta^1(K)$ and $q \in V_\beta^1(K)$. Integrating by parts in (37), we get

$$\int_K (u(sv - \Delta v + \nabla q) - p \nabla \cdot v) dx = 0 \quad \text{for all } v \in E_\beta^2(K) \cap \mathring{E}_\beta^1(K), \quad q \in V_\beta^1(K)$$

and, consequently, $(u, p) \in \ker A_\beta^*$. This proves (36).

By the first assertion, the kernel of $A_{-\gamma}$ is not trivial for $-\mu_2^+ - 1/2 < \gamma < -\lambda_1^+ - 3/2$. Consequently, by (36), the kernel of A_β^* is not trivial for $-\mu_2^+ - 1/2 < \beta < -\lambda_1^+ + 1/2$. \square

It remains to consider the case $\lambda_1^+ > \mu_2^+ - 1$. We construct some special solutions of the system $(s - \Delta)u + \nabla p = 0$, $\nabla \cdot u = 0$. Let $\nu(x)$ denote the distance of the point x from the boundary ∂K . Obviously, the function ν is positively homogeneous of degree 1. In the neighborhood $\nu(x) < \delta|x|$ of the boundary ∂K with sufficiently small δ , the function ν is two times continuously differentiable and satisfies the equality $|\nabla \nu| = 1$. Furthermore, the vector $\nabla \nu(x)$ is orthogonal to ∂K in any point $x \in \partial K$. For an arbitrary vector function u in the neighborhood $\nu(x) < \delta|x|$ of ∂K , we define

$$u_\nu = u \cdot \nabla \nu \quad \text{and} \quad u_\tau = u - u_\nu \nabla \nu.$$

Obviously $u_\tau \cdot \nabla \nu = 0$.

Lemma 2.16 *Let f and g be positively homogeneous functions of degree μ in the neighborhood $\nu < \delta r$ of the boundary ∂K . Furthermore, let p and q be polynomials of degree k and $k+1$, respectively, such that*

$$p'(\nu) - \sqrt{s}p(\nu) = \nu^k, \quad q''(\nu) - 2\sqrt{s}q'(\nu) = -\nu^k, \quad q(0) = 0.$$

Then the functions

$$U = e^{-\nu\sqrt{s}}(p(\nu)g\nabla\nu + q(\nu)f_\tau), \quad P = e^{-\nu\sqrt{s}}p(\nu)f_\nu,$$

satisfy the equalities

$$(s - \Delta)U + \nabla P = e^{-\nu\sqrt{s}}(\nu^k f + (p''(\nu) - 2k\nu^{k-1})g\nabla\nu + R)$$

and

$$\nabla \cdot U = e^{-\nu\sqrt{s}}(\nu^k g + p(\nu)\nabla \cdot (g\nabla\nu) + q(\nu)\nabla \cdot f_\tau),$$

in the neighborhood $\nu < \delta r$ of ∂K . Here R is a finite sum of terms of the form $\nu^j \phi$ with integer j , $0 \leq j \leq k+1$, and positively homogeneous functions ϕ of degree $\mu-1$ or $\mu-2$.

P r o o f. Since $\nabla v(\nu) = v'(\nu)\nabla\nu$, we obtain

$$\begin{aligned} \nabla \cdot U &= -\sqrt{s}e^{-\nu\sqrt{s}}p(\nu)g + e^{-\nu\sqrt{s}}\nabla \cdot (p(\nu)g\nabla\nu + q(\nu)f_\tau) \\ &= e^{-\nu\sqrt{s}}(\nu^k g + p(\nu)\nabla \cdot (g\nabla\nu) + q(\nu)\nabla \cdot f_\tau). \end{aligned}$$

Note that the functions $\nabla \cdot (g\nabla\nu)$ and $\nabla \cdot f_\tau$ are homogenous of degree $\mu-1$. Furthermore,

$$\nabla P = f_\nu \nabla(e^{-\nu\sqrt{s}}p(\nu)) + e^{-\nu\sqrt{s}}p(\nu)\nabla f_\nu = e^{-\nu\sqrt{s}}(\nu^k f_\nu \nabla\nu + p(\nu)\nabla f_\nu).$$

Using the equalities $(s - \Delta)e^{-\nu\sqrt{s}} = \sqrt{s}e^{-\nu\sqrt{s}}\Delta\nu$ and $\Delta p(\nu) = p''(\nu) + p'(\nu)\Delta\nu$, we get

$$\begin{aligned} (s - \Delta)U &= \sqrt{s}e^{-\nu\sqrt{s}}(p(\nu)g\nabla\nu + q(\nu)f_\tau)\Delta\nu \\ &\quad + 2\sqrt{s}e^{-\nu\sqrt{s}}\sum_{j=1}^3 \frac{\partial\nu}{\partial x_j} \frac{\partial}{\partial x_j}(p(\nu)g\nabla\nu + q(\nu)f_\tau) - e^{-\nu\sqrt{s}}\Delta(p(\nu)g\nabla\nu + q(\nu)f_\tau) \\ &= e^{-\nu\sqrt{s}}(\nu^k f_\tau + (2\sqrt{s}p'(\nu) - p''(\nu))g\nabla\nu + R), \end{aligned}$$

where

$$\begin{aligned} R &= p(\nu)\Delta(g\nabla\nu) + (\sqrt{s}p(\nu) - p'(\nu))(g\Delta\nu\nabla\nu + 2\sum_{j=1}^3 \frac{\partial\nu}{\partial x_j} \frac{\partial(g\nabla\nu)}{\partial x_j}) \\ &\quad + q(\nu)\Delta f_\tau + (\sqrt{s}q(\nu) - q'(\nu))(f_\tau\Delta\nu + 2\sum_{j=1}^3 \frac{\partial\nu}{\partial x_j} \frac{\partial f_\tau}{\partial x_j}). \end{aligned}$$

Obviously, $\Delta(g\nabla\nu)$ and Δf_τ are positively homogeneous of degree $\mu-2$, while $g\Delta\nu\nabla\nu$, $f_\tau\Delta\nu$, $\frac{\partial\nu}{\partial x_j} \frac{\partial(g\nabla\nu)}{\partial x_j}$ and $\frac{\partial\nu}{\partial x_j} \frac{\partial f_\tau}{\partial x_j}$ are positively homogeneous of degree $\mu-1$. This proves the lemma. \square

Lemma 2.17 *Suppose that $\lambda_1^+ > \mu_2^+ - 1$. Then $\dim \ker A_\beta^* \geq 2$ for $\mu_2^+ + 1/2 < \beta < \lambda_1^+ + 3/2$.*

P r o o f. By Lemma 2.12, the pair $(v, q) = (0, 1)$ belongs to the kernel of A_β^* . We construct another element of $\ker A_\beta^*$. Let ϕ be an eigenfunction of the pencil $\mathcal{N}(\lambda)$ corresponding to the eigenvalue $\lambda = \mu_2^+$. We start with the same pair (u_0, p_0) as in the proof of Lemma 2.8, i.e., $p_0 = r^{\mu_2^+} \phi(\omega)$ and

$$u_0(x) = v_0(x) - \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} v_{0,\tau}(x),$$

where $v_0 = -s^{-1} \nabla p_0$, $v_{0,\tau} = v_0 - v_{0,\nu} \nabla \nu$ for $\nu < \delta r$ and $v_{0,\nu} = v_0 \cdot \nabla \nu$ for $\nu < \delta r$. Here, χ is again a smooth cut-off function, $\chi = 1$ in $(0, \delta/2)$, $\chi = 0$ in (δ, ∞) . Then $u_0 = 0$ on ∂K and

$$(s - \Delta)u_0 + \nabla p_0 = \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} f_0 + F_0, \quad \nabla \cdot u_0 = \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} g_0 + G_0$$

in K , where

$$f_0 = -\sqrt{s} v_{0,\tau} \Delta \nu - 2\sqrt{s} \sum_{j=1}^3 \frac{\partial \nu}{\partial x_j} \frac{\partial v_{0,\tau}}{\partial x_j}, \quad g_0 = -\nabla \cdot v_{0,\tau}$$

and

$$F_0 = \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} \Delta v_{0,\tau} + \left[\Delta, \chi\left(\frac{\nu}{r}\right)\right] e^{-\nu\sqrt{s}} v_{0,\tau}, \quad G_0 = -e^{-\nu\sqrt{s}} v_{0,\tau} \cdot \nabla \chi\left(\frac{\nu}{r}\right)$$

(see (29) and (30)). Obviously, the functions f_0 and g_0 are positively homogeneous of degree $\mu_2^+ - 2$. Let η be a smooth function on \bar{K} such that $\eta(x) = 0$ for $|x| < 1$ and $\eta(x) = 1$ for $|x| > 2$. Then

$$\eta u_0 \in E_{-\beta}^2(K), \quad \eta p_0 \in V_{-\beta}^1(K)$$

since $\beta > \mu_2^+ + 1/2$. Furthermore, $\eta F_0 \in V_\gamma^0(K)$ and $\eta G_0 \in X_\gamma^1(K)$ for arbitrary $\gamma < \frac{3}{2} - \mu_2^+$.

We define

$$w_1 = \frac{1}{2\sqrt{s}} \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} (2g_0 \nabla \nu - \nu f_{0,\tau}), \quad q_1 = \frac{1}{\sqrt{s}} \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} f_{0,\nu}.$$

By Lemma 2.16,

$$(s - \Delta)w_1 + \nabla q_1 = -\chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} f_0 + F_1$$

and

$$\nabla \cdot w_1 = \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} (-g_0 + g_1 + \nu g_2) + G_1$$

where $g_1 = s^{-1/2} \nabla \cdot (g_0 \nabla \nu)$ and $g_2 = -\frac{1}{2} s^{-1/2} \nabla \cdot f_{0,\tau}$. Here, F_1 is a sum of terms of the form

$$\chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} \Phi \quad \text{or} \quad \chi^{(k)}\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} \Psi, \quad (38)$$

where Φ is positively homogeneous of degree $\leq \mu_2^+ - 3$ and Ψ is positively homogeneous of degree $\leq \mu_2^+ - 2$, while G_1 is a sum of terms of the form (38), where Φ is positively homogeneous of degree $\leq \mu_2^+ - 4$. Consequently, $\eta F_1 \in V_\gamma^0(K)$ and $\eta G_1 \in X_\gamma^1(K)$ for arbitrary $\gamma < \frac{3}{2} - \mu_2^+$. Obviously, $w_{1,\tau} = 0$ and $w_{1,\nu} = s^{-1/2} g_0$ on ∂K . Since g_0 is positively homogeneous of degree $\mu_2^+ - 2$, there exists a solution of the problem

$$\Delta p_1 = 0 \quad \text{in } K, \quad \frac{\partial p_1}{\partial n} = s w_{1,\nu} = s^{1/2} g_0 \quad \text{on } \partial K \setminus \{0\}$$

which has the form $p_1 = r^{\mu_2^+ - 1} \psi(\omega)$ if $\mu_2^+ - 1$ is not an eigenvalue or $p_1 = r^{\mu_2^+ - 1} (\psi_1(\omega) + \psi_2(\omega) \log r)$ if $\mu_2^+ - 1$ is an eigenvalue of the pencil $\mathcal{N}(\lambda)$ (see, e. g., [8, Lemma 6.1.13]). We suppose for simplicity that $\mu_2^+ - 1$ is not an eigenvalue of the pencil $\mathcal{N}(\lambda)$ and set $v_1 = -s^{-1} \nabla p_1$ and

$$u_1(x) = v_1(x) - \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} v_{1,\tau}.$$

Then $(s - \Delta)u_1 + \nabla p_1 = F_2$ and

$$\nabla \cdot u_1 = -\chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} \nabla \cdot v_{1,\tau} + G_2$$

where F_2 is a sum of terms of the form (38) with positively homogeneous functions Φ and Ψ of degree $\leq \mu_2^+ - 3$ and degree $\leq \mu_2^+ - 2$, respectively, and G_2 is a sum of terms of the form (38) with positively

homogeneous functions Φ of degree $\leq \mu_2^+ - 4$. In particular, $\eta F_2 \in V_\gamma^0(K)$ and $\eta G_2 \in X_\gamma^1(K)$ for arbitrary $\gamma < \frac{3}{2} - \mu_2^+$. Obviously, $u_{1,\tau} = 0$ and $u_{1,\nu} = v_{1,\nu} = -s^{-1} \nabla p_1 \cdot \nabla \nu = -w_{1,\nu}$ on $\partial K \setminus \{0\}$. This means that $u_1 + w_1 = 0$ on $\partial K \setminus \{0\}$.

In the last step, we consider the function

$$w_2 = s^{-1/2} \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} \left((g_1 - \nabla \cdot v_{1,\tau}) \nabla \nu + (\nu + s^{-1/2}) g_2 \nabla \nu \right).$$

Using Lemma 2.16, we conclude that

$$(s - \Delta) w_2 = F_3 \quad \text{and} \quad \nabla \cdot w_2 = -\chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} (g_1 - \nabla \cdot v_{1,\tau}) + \nu g_2 + G_3,$$

where $\eta F_3 \in V_\gamma^0(K)$ and $\eta G_3 \in X_\gamma^1(K)$ for arbitrary $\gamma < \frac{3}{2} - \mu_2^+$. Furthermore, $w_{2,\tau} = 0$ and $w_{2,\nu} = s^{-1/2} (g_1 - \nabla \cdot v_{1,\tau}) + s^{-1} g_2$ on $\partial K \setminus \{0\}$. Since the functions g_1, g_2 and $\nabla \cdot v_{1,\tau}$ are positively homogeneous of degree $\mu_2^+ - 3$, there exists a solution p_2 of the problem

$$\Delta p_2 = 0 \quad \text{in } K, \quad \frac{\partial p_2}{\partial n} = s^{1/2} (g_1 - \nabla \cdot v_{1,\tau}) + g_2 \quad \text{on } \partial K \setminus \{0\}$$

which has the form $p_2 = r^{\mu_2^+ - 2} \psi(\omega)$ if $\mu_2^+ - 1$ or $p_2 = r^{\mu_2^+ - 2} (\psi_1(\omega) + \psi_2(\omega) \log r)$. We set $v_2 = -s^{-1} \nabla p_2$ and

$$u_2(x) = v_2(x) - \chi\left(\frac{\nu}{r}\right) e^{-\nu\sqrt{s}} v_{2,\tau}(x).$$

Then $\eta(s - \Delta)u_2 + \nabla p_2 \in V_\gamma^0(K)$ and $\eta \nabla \cdot u_2 \in X_\gamma^1(K)$ for arbitrary $\gamma < \frac{3}{2} - \mu_2^+$. Furthermore $u_2 = -w_2$ on $\partial K \setminus \{0\}$.

We define

$$U = u_0 + u_1 + u_2 + w_1 + w_2 \quad \text{and} \quad P = p_0 + p_1 + p_2.$$

Since $u_0, u_1 + w_1$ and $u_2 + w_2$ are zero on the boundary, the function U also vanishes on $\partial K \setminus \{0\}$. Furthermore, as was show above, we have $\eta(s - \Delta)U + \eta P \in V_\gamma^0(K)$ and $\eta \nabla \cdot U \in X_\gamma^1(K)$ for arbitrary $\gamma < \frac{3}{2} - \mu_2^+$. Since $\nabla \eta = 0$ outside the region $1 < |x| < 2$, we conclude from this that

$$(s - \Delta)(\eta U) + \nabla(\eta P) \in V_\gamma^0(K), \quad \nabla \cdot (\eta U) \in X_\gamma^1(K)$$

for arbitrary $\gamma < \frac{3}{2} - \mu_2^+$. By the condition of the lemma on λ_1^+ and μ_2^+ , the number γ can be chosen such that $\frac{1}{2} - \lambda_1^+ < \gamma < \min(\frac{1}{2}, 2 - \beta)$. In particular, $\gamma > -\frac{1}{2}$. By Theorem 2.2, there exists a solution $(u_3, p_3) \in E_\gamma^2(K) \times V_\gamma^1(K)$ of the problem

$$(s - \Delta) u_3 + \nabla p_3 = (s - \Delta)(\eta U) + \nabla(\eta P), \quad \nabla \cdot u_3 = \nabla \cdot (\eta U) \quad \text{in } K$$

with the Dirichlet condition $u_3 = 0$ on $\partial K \setminus \{0\}$. This means that $u = \eta U - u_3$ and $p = \eta P - p_3$ satisfy the equations $(s - \Delta)u + \nabla p = 0$, $\nabla \cdot u = 0$ in K and the boundary condition $u = 0$ on $\partial K \setminus \{0\}$. Obviously, $u \neq 0$ since $\eta U \notin E_\gamma^2(K)$. Furthermore,

$$\eta(U, P) \in E_{-\beta}^2(K) \times V_{-\beta}^1(K) \subset E_{-\beta}^0(K) \times (X_\beta^1(K))^*.$$

The same is true for (u_3, p_3) . Indeed, we have $u_3 \in E_{\gamma-2}^0(K) \cap E_\gamma^0(K) \subset E_{-\beta}^0(K)$, since $\gamma - 2 < -\beta < -\frac{1}{2} < \gamma$, and from $p_3 \in V_\gamma^1(K)$ it follows that $\eta p_3 \in V_{-\beta}^1(K)$ (since $-\beta < \gamma$) and $(1 - \eta)p_3 \in V_{1-\beta}^0(K) \subset (V_\beta^1(K))^*$ (since $1 - \beta > \gamma - 1$). Thus, $p_3 \in V_{-\beta}^1(K) + (V_\beta^1(K))^* = (X_\beta^1(K))^*$. This proves that $(u, p) \in E_{-\beta}^0(K) \times (X_\beta^1(K))^*$ is an element of the kernel of A_β^* . The proof is complete. \square

2.9 An estimate for arbitrary $s \neq 0$, $\text{Re } s \geq 0$

We consider the problem (7), where s is an arbitrary complex parameter $\neq 0$ with nonnegative real part. As a consequence of Theorems 2.2 and 2.3, we obtain the following result.

Theorem 2.4 *Suppose that $s \neq 0$ and $\text{Re } s \geq 0$. Furthermore, we assume that $f \in V_\beta^0(K)$ and $g \in X_\beta^1(K)$, where $-\lambda_1^+ + 1/2 < \beta < \min(\mu_2^+ + 1/2, \lambda_1^+ + 3/2)$ and $\beta \neq 1/2$. In the case $\beta > 1/2$, we assume in addition that $\int_K g(x) dx = 0$. Then there exists a uniquely determined solution $(u, p) \in E_\beta^2(K) \times V_\beta^1(K)$ of the problem (7). This solution satisfies the estimate*

$$\|u\|_{V_\beta^2(K)} + |s| \|u\|_{V_\beta^0(K)} + \|p\|_{V_\beta^1(K)} \leq c \left(\|f\|_{V_\beta^0(K)} + \|g\|_{V_\beta^1(K)} + |s| \|g\|_{(V_{-\beta}^1(K))^*} \right)$$

with a constant c independent of f, g and s .

P r o o f. The pair (u, p) is a solution of the problem (7) if and only if

$$v(x) = u(|s|^{-1/2}x) \quad \text{and} \quad q(x) = |s|^{-1/2}p(|s|^{-1/2}x)$$

satisfy the equations

$$\frac{s}{|s|}v - \Delta v + \nabla q = F, \quad -\nabla \cdot v = G \quad \text{in } K, \quad v|_{\partial K \setminus \{0\}} = 0, \quad (39)$$

where $F(x) = |s|^{-1}f(|s|^{-1/2}x)$ and $G(x) = |s|^{-1/2}g(|s|^{-1/2}x)$. By Theorems 2.2 and 2.3, the boundary value problem for the system (39) has a uniquely determined solution $(v, q) \in E_\beta^2(K) \times V_\beta^1(K)$ satisfying the estimate

$$\|v\|_{E_\beta^2(K)}^2 + \|q\|_{V_\beta^1(K)}^2 \leq c \left(\|F\|_{V_\beta^0(K)}^2 + \|G\|_{V_\beta^1(K)}^2 + \|G\|_{(V_\beta^1(K))^*}^2 \right).$$

One easily verifies that

$$\|F\|_{V_\beta^0(K)}^2 + \|G\|_{V_\beta^1(K)}^2 + \|G\|_{(V_\beta^1(K))^*}^2 = |s|^{\beta-1/2} \left(\|f\|_{V_\beta^0(K)}^2 + \|g\|_{V_\beta^1(K)}^2 + |s|^2 \|g\|_{(V_\beta^1(K))^*}^2 \right)$$

and

$$\begin{aligned} \|v\|_{E_\beta^2(K)}^2 + \|q\|_{V_\beta^1(K)}^2 &\geq \|v\|_{V_\beta^2(K)}^2 + \|v\|_{V_\beta^0(K)}^2 + \|q\|_{V_\beta^1(K)}^2 \\ &= |s|^{\beta-1/2} \left(\|u\|_{V_\beta^2(K)}^2 + |s|^2 \|u\|_{V_\beta^0(K)}^2 + \|p\|_{V_\beta^1(K)}^2 \right). \end{aligned}$$

This proves the theorem. \square

3 The time-dependent problem in K

Now, we consider the problem (1), (2). Using the last theorem, we can easily show that this problem has a uniquely determined solution in a certain weighted Sobolev space.

3.1 Weighted Sobolev spaces in $K \times \mathbb{R}_+$

Let $Q = K \times \mathbb{R}_+ = K \times (0, \infty)$. We denote by $W_\beta^{2l,l}(Q)$ the weighted Sobolev space of all functions $u = u(x, t)$ on Q with finite norm

$$\|u\|_{W_\beta^{2l,l}(Q)} = \left(\int_0^\infty \sum_{k=0}^l \|\partial_t^k u(\cdot, t)\|_{V_\beta^{2l-2k}(K)}^2 dt \right)^{1/2}.$$

In particular, $W_\beta^{0,0}(Q) = L_2(\mathbb{R}_+, V_\beta^0(K))$ and $W_\beta^{2,1}(Q)$ is the set of all $u \in L_2(\mathbb{R}_+, V_\beta^2(K))$ such that $\partial_t u \in L_2(\mathbb{R}_+, V_\beta^0(K))$. The space $\mathring{W}_\beta^{2l,l}(Q)$ is the subspace of all $u \in W_\beta^{2l,l}(Q)$ satisfying the condition $\partial_t^k u|_{t=0} = 0$ for $x \in K$, $k = 0, \dots, l-1$. Note that $\partial_t^k u(\cdot, 0) \in V_\beta^{2l-2k-1}(K)$ for $u \in W_\beta^{2l,l}(Q)$, $k = 0, \dots, l-1$ (see [5, Proposition 3.1]). By [5, Proposition 3.4], the Laplace transform realizes an isomorphism from $\mathring{W}_\beta^{2l,l}(Q)$ onto the space H_β^{2l} of all holomorphic functions $\tilde{u}(x, s)$ for $\text{Re } s > 0$ with values in $E_\beta^{2l}(K)$ and finite norm

$$\|\tilde{u}\|_{H_\beta^{2l}} = \sup_{\gamma > 0} \left(\int_{-\infty}^{+\infty} \sum_{k=0}^l |s|^{2k} \|\tilde{u}(\cdot, \gamma + i\tau)\|_{V_\beta^{2l-2k}(K)}^2 d\tau \right)^{1/2}.$$

The proof for the analogous result in nonweighted spaces can be found in [1, Theorem 8.1].

3.2 Existence and uniqueness of solutions

As a consequence of Theorem 2.4, we obtain the following assertion.

Theorem 3.1 Suppose that $f \in L_2(\mathbb{R}_+, V_\beta^0(K))$, $g \in L_2(\mathbb{R}_+, V_\beta^1(K))$ and $\partial_t g \in L_2(\mathbb{R}_+, (V_{-\beta}^1(K))^*)$, where $-\lambda_1^+ + 1/2 < \beta < \min(\mu_2^+ + 1/2, \lambda_1^+ + 3/2)$ and $\beta \neq 1/2$. In the case $\beta > 1/2$, we assume in addition that $\int_K g(x, t) dx = 0$ for almost all t . Then there exists a uniquely determined solution $(u, p) \in W_\beta^{2,1}(Q) \times L_2(\mathbb{R}_+, V_\beta^1(K))$ of the problem (1), (2) satisfying the estimate

$$\|u\|_{W_\beta^{2,1}(Q)} + \|p\|_{L_2(\mathbb{R}_+, V_\beta^1(K))} \leq c \left(\|f\|_{W_\beta^{0,0}(Q)} + \|g\|_{L_2(\mathbb{R}_+, V_\beta^1(K))} + \|\partial_t g\|_{L_2(\mathbb{R}_+, (V_{-\beta}^1(K))^*)} \right)$$

with a constant c independent of f, g .

P r o o f. Let $\tilde{f} \in H_\beta^0$ and \tilde{g} be the Laplace transforms (with respect to the variable t) of f and g . For arbitrary $s \neq 0$, $\operatorname{Re} s \geq 0$, there exists a uniquely determined solution $\tilde{u}(\cdot, s), \tilde{p}(\cdot, s) \in E_\beta^2(K) \times V_\beta^1(K)$ of the problem (3) satisfying the estimate

$$\begin{aligned} & \|\tilde{u}(\cdot, s)\|_{V_\beta^2(K)}^2 + |s|^2 \|\tilde{u}(\cdot, s)\|_{V_\beta^0(K)}^2 + \|\tilde{p}(\cdot, s)\|_{V_\beta^1(K)}^2 \\ & \leq c \left(\|\tilde{f}(\cdot, s)\|_{V_\beta^0(K)}^2 + \|\tilde{g}(\cdot, s)\|_{V_\beta^1(K)}^2 + |s|^2 \|\tilde{g}(\cdot, s)\|_{(V_{-\beta}^1(K))^*}^2 \right) \end{aligned}$$

with a constant c independent of s . Integrating over the line $\operatorname{Re} s = \gamma$ and taking the supremum with respect to $\gamma > 0$, we get the assertion of the theorem. \square

Furthermore, we can deduce the following regularity result from Lemma 2.13.

Theorem 3.2 Suppose that $(u, p) \in W_\beta^{2,1}(Q) \times L_2(\mathbb{R}_+, V_\beta^1(K))$ is a solution of the problem (1), (2), where

$$f \in L_2(\mathbb{R}_+, V_\beta^0(K)) \cap L_2(\mathbb{R}_+, V_\gamma^0(K)),$$

$$g \in L_2(\mathbb{R}_+, V_\beta^1(K)) \cap L_2(\mathbb{R}_+, V_\gamma^1(K)), \quad \partial_t g \in L_2(\mathbb{R}_+, (V_{-\beta}^1(K))^*) \cap L_2(\mathbb{R}_+, (V_{-\gamma}^1(K))^*),$$

$1/2 - \lambda_1^+ < \beta, \gamma < \min(\mu_2^+ + 1/2, \lambda_1^+ + 3/2)$, $\beta \neq 1/2$ and $\gamma \neq 1/2$. In the case $\gamma > 1/2$ we assume in addition that $\int_K g(x, t) dx = 0$ for almost all t . Then $u \in W_\gamma^{2,1}(Q)$ and $p \in L_2(\mathbb{R}_+, V_\gamma^1(K))$.

P r o o f. By Theorem 3.1, there exists a uniquely determined solution $(v, q) \in W_\gamma^{2,1}(Q) \times L_2(\mathbb{R}_+, V_\gamma^1(K))$ of the problem (1), (2). The Laplace transforms $(\tilde{u}(x, s), \tilde{p}(x, s))$ and $(\tilde{v}(x, s), \tilde{q}(x, s))$ belong to the spaces $E_\beta^2(K) \times V_\beta^1(K)$ and $E_\gamma^2(K) \times V_\gamma^1(K)$ for almost all s in the half-plane $\operatorname{Re} s \geq 0$. Using Lemma 2.13, we obtain $(\tilde{u}, \tilde{p}) = (\tilde{v}, \tilde{q})$ and, consequently, $(u, p) = (v, q)$. \square

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